PERGAMON

Topology 42 (2003) 1003–1063

TOPOLOGY

www.elsevier.com/locate/top

A long exact sequence for symplectic Floer cohomology

Paul Seidel

Department of Mathematics, Imperial College, Huxley Building, 180 Queens Gate, SW7 2BE London, UK

Received 30 May 2001; accepted 27 February 2002

Let (M^{2n}, ω, α) be a compact symplectic manifold with contact type boundary: α is a contact one-form on ∂M which satisfies $d\alpha = \omega|\partial M$ and makes ∂M convex. Assume in addition that $[\omega, \alpha] \in H^2(M, \partial M; \mathbb{R})$ is zero, so that α can be extended to a one-form θ on M satisfying $d\theta = \omega$. After fixing such a θ once and for all, one can talk about exact Lagrangian submanifolds in M. The Floer cohomology of two such submanifolds is comparatively easy to define, since the corresponding action functional has no periods, so that bubbling is impossible. The aim of this paper is to prove the following result, which was announced in [28] (with an additional assumption on $c_1(M)$ that has been removed in the meantime).

Theorem 1. *Let L be an exact Lagrangian sphere in M together with a preferred diffeomorphism $f : S^n \to L$. One can associate to it an exact symplectic automorphism of M, the Dehn twist $\tau_L = \tau_{(L,[f])}$. For any two exact Lagrangian submanifolds $L_0, L_1 \subset M$, there is a long exact sequence of Floer cohomology groups*

$$\begin{tikzcd} HF(\tau_L(L_0), L_1) \arrow[rr] & & HF(L_0, L_1) \arrow[dl] \\ & HF(L, L_1) \otimes HF(L_0, L). \arrow[ul] & \end{tikzcd} \tag{0.1}$$

The original inspiration for this came from the exact sequence in Donaldson–Floer theory [3], which can be translated into symplectic geometry using various versions, proved [5] and unproved, of the Atiyah–Floer conjecture. That line of thought should have a Seiberg–Witten sibling, starting from [4], but the corresponding Atiyah–Floer type relationships are only just beginning to be understood [23,18]. In any case, the exact sequences obtained from such speculations differ somewhat in generality from that stated above. This reflects the fact that the first motivation has been largely superseded by different ones, coming from mirror symmetry. Kontsevich's homological mirror conjecture [12] for Calabi–Yau varieties implies a relation between symplectic automorphisms and

E-mail address: p.seidel@ic.ac.uk (P. Seidel).

PII: S0040-9383(02)00028-9

self-equivalences of derived categories of coherent sheaves; see survey [2] or papers [29,9]. A particular class of self-equivalences, “twist functors along spherical objects”, is expected to correspond to Dehn twists. By definition, twist functors give rise to an exact sequence of the same form as (0.1), with Floer cohomology replaced by Ext-groups, so that the expected correspondence fits in well with our result. With respect to the whole of Kontsevich’s conjecture, this is a rather peripheral issue. To see the exact sequence take on a more important role, one has to pass to a related context, namely mirror symmetry for Fano varieties. The derived categories of coherent sheaves on such varieties are often generated by exceptional collections, which are subject to transformations called mutations [22]. The mirror dual notion is that of distinguished basis of vanishing cycles, which is well-known in Picard–Lefschetz theory. A rigorous connection between the two concepts is established in [28, Theorem 3.3] whose proof relies strongly on Theorem 1; at present, this would seem to be its main application. We conclude our discussion with some more concrete remarks about the statement of the theorem:

(i) In this paper, Floer cohomology groups are treated as ungraded groups. One can of course assume that L, L_0, L_1 are oriented, and then the groups become $\mathbb{Z}/2$-graded. Different conventions are in use, but if one adopts that in which the Euler characteristic of Floer cohomology is $(-1)^{n(n+1)/2}$ times the intersection number, the degrees mod two of the maps in (0.1) are

$$\begin{tikzcd} {} \arrow[rr, "0"] & & {} \arrow[dl, "n"] \\ & {} \arrow[ul, "1-n"] & \end{tikzcd} \tag{0.2}$$

Under more restrictive assumptions, one can introduce $\mathbb{Z}$-gradings. There are several roughly equivalent ways of doing this; we adopt the approach of [12,27], in which one fixes a trivialization of the bicanonical bundle K_M^2, thus establishing a notion of “graded Lagrangian submanifold”. Suppose that preferred gradings have been chosen for L, L_0, L_1. Through the grading of τ_L itself, this induces a grading of $\tau_L(L_0)$. All Floer cohomology groups in the exact sequence are then canonically $\mathbb{Z}$-graded, and the degrees of the maps are as in (0.2). This is not difficult to show, it just requires a few Maslov index computations.

(ii) We use Floer cohomology with $\mathbb{Z}/2$-coefficients. Inspection of the discussion of coherent orientation in [8] suggests that at least when L_0, L_1 are spin, the exact sequence should exist with $\mathbb{Z}$-coefficients (with the $\otimes$ replaced by the cohomology of the underlying tensor product of cochain groups, to avoid Künneth terms). However, I have not checked all the details.

(iii) The assumption $[\omega, \alpha] = 0$ is the main limitation of the theorem as it stands. We have adopted this “exact” framework with a view to the application in [28], and also because it simplifies a number of technical issues, thereby hopefully allowing the basic ideas to stand out. Almost the same proof goes through in a few other cases, such as when suitable “monotonicity” conditions hold. On the other hand, a considerable amount of work remains to be done to extend the exact sequence to the most general situation where one would want to have it; a version for closed manifolds with $c_1(M) = 0$ seems particularly desirable.

(iv) The map $\nwarrow$ in (0.1) is obtained by composing the canonical isomorphism $HF(L_0, L) \cong HF(\tau_L(L_0), L)$ which exists because $\tau_L(L) = L$, with a pair-of-pants product (a.k.a. Donaldson product). In the spirit of [19], this can be seen as a kind of relative Gromov invariant. A more general version of the same formalism, involving pseudo-holomorphic sections of fibrations with singularities, yields the second map $\to$. In contrast, $\swarrow$ appears as connecting map in our construction, and is therefore defined only indirectly. A symmetry consideration using the duality $\mathrm{HF}(L_0, L_1) \cong \mathrm{HF}(L_1, L_0)^\vee$ suggests that this map should actually be a pair-of-pants coproduct. That is in fact true, but we will not prove it here.

The exposition in the body of the paper follows a slightly indirect course, in that we try to familiarize the reader with each ingredient separately, before they all get mixed up into the main argument. There is even a small amount of material which is not necessary for our immediate purpose, but which is closely related and useful for further development. Thus, the whole of Chapter 1 is elementary symplectic geometry, concentrating on topics related to Picard–Lefschetz theory; Chapter 2 deals with pseudo-holomorphic curves, which means setting up the relative invariants mentioned above, and introducing certain techniques for partially computing them based on symplectic curvature; and the only the third chapter addresses the actual proof.

1. Dehn twists, and all that

This chapter takes a look at basic Picard–Lefschetz theory from the symplectic viewpoint. It has been known since Arnold's note [1] that such a viewpoint makes sense, and it has been used for various purposes, see, e.g. [26,11]. Still, the present paper seems to be the most systematic attempt at an exposition so far. The reader may be surprised by the exactness assumptions built into our framework. As far as the elementary theory is concerned, there is no need to make such assumptions. However, they greatly simplify the pseudo-holomorphic theory to be introduced later on, and in order to keep the set-up coherent, we have chosen to impose them from the start.

1.1. Exact symplectic geometry and fibrations

By an exact symplectic manifold we mean a compact manifold M with boundary, together with a symplectic form ω and a one-form θ satisfying $\mathrm{d}\theta = \omega$, such that $\theta|\partial M$ is a contact one-form and makes ∂M convex. An isomorphism of exact symplectic manifolds is a diffeomorphism $\phi : M \to M'$ which is symplectic, satisfies $\phi^*\theta' = \theta$ in some neighbourhood of ∂M, and such that $[\phi^*\theta' - \theta] \in H^1(M, \partial M; \mathbb{R})$ is zero. This means that there is a unique function $K_\phi \in C_c^\infty(M \setminus \partial M, \mathbb{R})$ such that $\phi^*\theta' = \theta + \mathrm{d}K_\phi$. We denote by $\mathrm{Symp}^{\mathrm{e}}(M)$ the group of those exact symplectic automorphisms of M which are the identity near ∂M. Its Lie algebra consists of vector fields X such that $\omega(\cdot, X) = \mathrm{d}H$ for some H which vanishes near ∂M, and is thus identified with $C_c^\infty(M \setminus \partial M, \mathbb{R})$.

An exact Lagrangian submanifold in M is a pair consisting of a Lagrangian submanifold $L \subset M$ (always assumed to be disjoint from ∂M) and a function K_L on it such that $\mathrm{d}K_L = \theta|L$. The image of L under an isomorphism $\phi : M \to M'$ of exact symplectic manifolds is again an exact Lagrangian submanifold, in a canonical way; the associated function is

$$K_{\phi(L)} = (K_L + K_\phi|L) \circ \phi^{-1}. \tag{1.1}$$

In particular, $\mathrm{Symp}^{\mathrm{e}}(M)$ acts on the set of exact Lagrangian submanifolds. A special situation which will occur later on is that $\phi \in \mathrm{Symp}^{\mathrm{e}}(M)$ satisfies $\phi(L) = L$ in the ordinary sense. Then (supposing L to be connected) $K_{\phi(L)} = K_L + c$ for some constant $c = K_\phi|L$, which need not be zero. This means that ϕ may not map L to itself as an exact Lagrangian submanifold, instead "shifting" it by some amount.

A notion of fibre bundle suitable for exact symplectic geometry is as follows. Let S be a smooth connected manifold, possibly with boundary (one could also allow corners), and $\pi: E \to S$ a differentiable fibre bundle whose fibres are compact manifolds with boundary. Write $\partial_{\mathrm{h}}E \subset E$ for the union of the boundaries of all the fibres. If the boundary of S is empty, $\partial_{\mathrm{h}}E = \partial E$; otherwise ∂E has another face $\partial_{\mathrm{v}}E = \pi^{-1}(\partial S)$, and the two faces meet at a codimension two corner. An exact symplectic fibration is such an (E,π) equipped with $\Omega \in \Omega^2(E)$ and $\Theta \in \Omega^1(E)$, satisfying $\mathrm{d}\Theta = \Omega$, such that each fibre E_z with $\omega_z = \Omega|E_z$ and $\theta_z = \Theta|E_z$ is an exact symplectic manifold. There is an additional condition of triviality near $\partial_{\mathrm{h}}E$, by which we mean the following: choose some $z \in S$ and consider the trivial fibration $\tilde{\pi}: \tilde{E} = S \times E_z \to S$, with the forms $\tilde{\Omega}, \tilde{\Theta}$ which are pullbacks of ω_z, θ_z. There should be a fibrewise diffeomorphism

$$\begin{tikzcd} N \arrow[rr] \arrow[dr, "\pi"'] & & \tilde{N} \arrow[dl, "\tilde{\pi}"] \\ & S & \end{tikzcd} \tag{1.2}$$

between neighbourhoods $N \subset E$, $\tilde{N} \subset \tilde{E}$ of $\partial_{\mathrm{h}}E$, resp. $\partial_{\mathrm{h}}\tilde{E}$, which maps $\partial_{\mathrm{h}}E$ to $\partial_{\mathrm{h}}\tilde{E}$, equals the identity on the fibre over z, and sends Ω, Θ to $\tilde{\Omega}, \tilde{\Theta}$. Note that the choice of z and the diffeomorphism are not considered to be part of the data defining an exact symplectic fibration; only their existence is assumed.

Lemma 1.1. *Take a point $z \in S$ and a chart $\psi: U \to S$, with $U \subset \mathbb{R}^k$ a contractible neighbourhood of* 0, *such that $\psi(0) = z$. Then there is a trivialization $\Psi: U \times E_z \to \psi^*E$, such that $\Psi \mid \{0\} \times E_z = \mathrm{id}$ and*

$$\Psi^*\Theta = \theta_z + \sum_{i=1}^{k} H_i \mathrm{d}t_i + \mathrm{d}R.$$

Here $H_1, \ldots, H_k$, $R \in C^\infty(U \times E_z, \mathbb{R})$ are functions which vanish near $U \times \partial E_z$, and t_i the coordinates on $\mathbb{R}^k$. Moreover, the difference between any two such trivializations Ψ_1, Ψ_2 is a map $\Psi_2^{-1} \circ \Psi_1: (U, 0) \to (\mathrm{Symp}^{\mathrm{e}}(E_z), \mathrm{id})$.

The proof is by a standard argument involving Moser's Lemma. The result means first of all that any exact symplectic fibration (E,π) has a structure of $\mathrm{Symp}^{\mathrm{e}}(E_z)$-fibre bundle, where E_z is any fibre. In addition to that, E carries a preferred connection, which gives rise to canonical parallel transport maps $\rho_c: E_{c(a)} \to E_{c(b)}$ over smooth paths $c: [a;b] \to S$; these are exact symplectic isomorphisms, and lie in $\mathrm{Symp}^{\mathrm{e}}(E_{c(a)})$ if c is closed. To define the preferred connection, one can use a local trivialization as before and set $A = \sum_i H_i \,\mathrm{d}t_i$, which is a one-form on U with values in $C_c^\infty(E_z \setminus \partial E_z, \mathbb{R})$. It is easy to check that this transforms in the proper way. A more intrinsic approach is to observe that $TE_x = TE_x^{\mathrm{h}} \oplus TE_x^{\mathrm{v}}$ splits into a horizontal and a vertical piece, given by $TE_x^{\mathrm{v}} = \ker(D\pi_x)$

and

$$TE_x^{\mathrm{h}} = \{X \in TE_x : \Omega(X,\cdot)|TE_x^{\mathrm{v}} = 0\}. \tag{1.3}$$

Each $Z \in TS_z$ has a unique lift $Z^{\mathrm{h}} \in C^\infty(TE^{\mathrm{h}}|E_z)$, and these vectors define the same connection as before (conversely, one can show that given a $\mathrm{Symp}^{\mathrm{e}}(M)$-fibre bundle and a compatible connection, one can equip its total space with the structure of an exact symplectic fibration). In a local trivialization as in Lemma 1.1, the curvature of the connection is

$$F_A = \sum_{i<j} \left(-\frac{\partial H_i}{\partial t_j} + \frac{\partial H_j}{\partial t_i} - \omega_z(X_i, X_j) \right) \mathrm{d}t_i \wedge \mathrm{d}t_j, \tag{1.4}$$

where X_i is the family of Hamiltonian vector fields on E_z corresponding to $H_i(t,\cdot)$. If one takes the more intrinsic view, the curvature is a two-form on S with values in functions on the fibres, and is given by $(Z_1, Z_2) \mapsto \Omega(Z_1^{\mathrm{h}}, Z_2^{\mathrm{h}})$. In the case where the base S is an oriented surface, we say that (E,π) is nonnegatively curved if for any oriented chart ψ and trivialization Ψ, the function in front of $\mathrm{d}t_1 \wedge \mathrm{d}t_2$ in (1.4) is nonnegative; or equivalently, if $\Omega|TE^{\mathrm{h}}$ is nonnegative for the induced orientation of TE^{h}. To see what this means, consider an exact symplectic fibration (E,π) over the closed unit disc $\bar{D}(1) \subset \mathbb{C}$, and the monodromy $\rho = \rho_{\partial\bar{D}(1)} : E_1 \to E_1$ around the boundary in positive sense. If the curvature is nonnegative, one can write ρ as time-one map of some (time-dependent) Hamiltonian on E_1 which vanishes near ∂E_1 and is $\leqslant 0$ everywhere.

Example 1.2. Let $E^\rho = \mathbb{R} \times M/(t,x) \sim (t-1, \rho(x))$ be the mapping torus of $\rho \in \mathrm{Symp}^{\mathrm{e}}(M)$. To make this into an exact symplectic fibration over $S^1 = \mathbb{R}/\mathbb{Z}$, one chooses a function $R_\rho \in C^\infty(\mathbb{R} \times M, \mathbb{R})$ such that $R_\rho(t-1, \rho(x)) = R_\rho(t,x) - K_\rho(x)$, and then sets $\Omega_{E^\rho} = \omega$, $\Theta_{E^\rho} = \theta + \mathrm{d}R_\rho$. The preferred connection is the one induced from the trivial connection on $\mathbb{R} \times M$, and its monodromy is ρ itself. It is easy to prove that all exact symplectic fibrations over a circle are of this form.

For future use, we make an observation on the compatibility of symplectic parallel transport with Lagrangian submanifolds. Let $c : [a;b] \to S$ be a smooth embedded path. Suppose that we have an exact Lagrangian submanifold in each fibre $E_{c(t)}$, depending smoothly on t; by this we mean a subbundle $Q \subset E|\mathrm{im}(c)$ such that each fibre $Q_{c(t)} \subset E_{c(t)}$ is Lagrangian, together with a $K_Q \in C^\infty(Q, \mathbb{R})$ whose restrictions $K_{Q_{c(t)}} = K_Q|Q_{c(t)}$ make the $Q_{c(t)}$ exact.

Lemma 1.3. *Assume that all the $Q_{c(t)}$ are connected. Then the following conditions are equivalent*:

(i) $\Omega|Q = 0$,
(ii) $\Theta|Q = \mathrm{d}K_Q + \pi^*\kappa_Q$ *for some* $\kappa_Q \in \Omega^1(\mathrm{im}(c))$,
(iii) *the maps* $\rho_{c|[s;s']} : E_{c(s)} \to E_{c(s')}$ *satisfy* $\rho_{c|[s;s']}(Q_{c(s)}) = Q_{c(s')}$ *for all* s, s'.

The proof is straightforward. To be precise, (iii) concerns $Q_{c(s)}$ as Lagrangian submanifolds only. Taking the functions into account and using (1.1) yields

$$K_{\rho_{c|[s;s']}(Q_{c(s)})} = K_{Q_{c(s')}} + \int_{c|[s;s']} \kappa_Q.$$

Hence $\rho_{c|[s;s']}(Q_{c(s)}) = Q_{c(s')}$ holds in the sense of exact Lagrangian submanifolds iff $\kappa_Q = 0$, or what is the same, $\Theta|Q = \mathrm{d}K_Q$.

Just like any kind of fibre bundle with connection, exact symplectic fibrations can be manipulated by cut-and-paste methods. As an example, take two oriented surfaces S^k, $k = 1,2$, and boundary circles $C^k \subset \partial S^k$; and let S be the surface obtained by identifying C^1, C^2 orientation-reversingly. Suppose that we have exact symplectic fibrations $(E^k, \pi^k, \Omega^k, \Theta^k)$ over S^k whose monodromies around C^k, taken in opposite senses, coincide. Then, assuming additionally that (E^k, π^k) is flat (has zero curvature) near C^k, one can construct from them an exact symplectic fibration (E, π) over S. It is maybe useful to give some details of this. To start, take oriented collars $\psi^1 : (-1; 0] \times \mathbb{R}/\mathbb{Z} \to S^1$, $\psi^2 : [0; 1) \times \mathbb{R}/\mathbb{Z} \to S^2$ around C^1, C^2, respectively, so that S can be defined by using $\psi^2 \circ (\psi^1)^{-1}$ to identify the two circles. Our main assumption is that there should be an exact symplectic isomorphism between the fibres of E^k over $\psi^k(0,0)$, say, which relates the monodromies around $\psi^k(\{0\} \times \mathbb{R}/\mathbb{Z})$. Because of flatness, an equivalent formulation is that there is some mapping torus E^ρ as in Example 1.2 and diffeomorphisms

$$\begin{array}{ccc} (-1;0] \cdot E^\rho & \xrightarrow{\Psi^1} & E^1 \\ \downarrow & & \downarrow{\scriptstyle \pi^1} \\ (-1;0] \cdot \mathbb{R}/\mathbb{Z} & \xrightarrow{\psi^1} & S^1 \end{array} \qquad \begin{array}{ccc} [0;1) \cdot E^\rho & \xrightarrow{\Psi^2} & E^2 \\ \downarrow & & \downarrow{\scriptstyle \pi^2} \\ [0;1) \cdot \mathbb{R}/\mathbb{Z} & \xrightarrow{\psi^2} & S^1 \end{array}$$

such that $(\Psi^k)^* \Omega^k = \Omega_{E^\rho}$ and $(\Psi^k)^* \Theta^k = \Theta_{E^\rho} + \mathrm{d}R^k$ for some functions R^k. Clearly, one can introduce modified forms $\tilde{\Theta}^k = \Theta^k - \mathrm{d}\tilde{R}^k$ with suitable functions $\tilde{R}^k$, in such a way that $(\Psi^k)^* \tilde{\Theta}^k = \Theta_{E^\rho}$ near $\{0\} \times E^\rho$. Gluing together the (E^k, π^k) along C^k via $\Psi^2 \circ (\Psi^1)^{-1}$ yields a smooth fibration (E, π) over S, and the $\Omega^k, \tilde{\Theta}^k$ match up to forms Ω, Θ on it, making it an exact symplectic fibration. Since we have not changed the symplectic connection, nonnegativity of the curvature of (E^k, π^k) implies the same for (E, π).

Remark 1.4. The flatness condition on (E^k, π^k) near C^k can be removed. Namely, suppose that it is not satisfied for $k = 1$. What one does then is to choose a function $g \in C^\infty([-1;0], \mathbb{R})$ such that $g(s) = s$ for s close to -1, $g(s) = 0$ for s close to 0, and $g'(s) \geqslant 0$ everywhere; and define a self-map p of the surface S^1 by $p(z) = z$ for $z \notin \mathrm{im}(\psi^1)$, $p(\psi^1(s,t)) = \psi^1(g(s), t)$. This collapses a small neighbourhood of C^1 onto that boundary circle, so if we replace (E^1, π^1) by its pullback under p, it becomes flat near C^1, and the previous construction goes through. It is noteworthy that this still preserves nonnegative curvature, because Dp has determinant $\geqslant 0$ everywhere.

The basic objects of Picard–Lefschetz theory are fibrations over surfaces, where the fibres are allowed to have certain particularly simple singularities. Let S be a connected oriented surface, possibly with boundary. An *exact Lefschetz fibration*[1] over S consists of data $(E, \pi, \Omega, \Theta, J_0, j_0)$ as follows. E is a $(2n+2)$-dimensional manifold whose boundary is the union of two faces $\partial_{\mathrm{h}} E$ and $\partial_{\mathrm{v}} E$, meeting at a codimension two corner. $\pi : E \to S$ is a proper map with $\partial_{\mathrm{v}} E = \pi^{-1}(\partial S)$ (so this may be empty), and such that both $\pi|\partial_{\mathrm{h}} E : \partial_{\mathrm{h}} E \to S$ and $\pi|\partial_{\mathrm{v}} E : \partial_{\mathrm{v}} E \to \partial S$ are smooth fibre bundles. π can

[1] This is called “exact Morse fibration” in [28]. The present terminology is more in line with general usage, since the notion is closely related to, even though not quite the same as, the symplectic Lefschetz fibrations considered by Donaldson, Gompf, and others.

have at most finitely many critical points, and no two may lie on the same fibre (moreover, because of the previous assumptions, they must lie in the interior of E). Denote by $E^{\mathrm{crit}} \subset E$, $S^{\mathrm{crit}} \subset S$ the set of critical points, resp. of critical values. J_0 is a complex structure on a neighbourhood of E^{crit}, and j_0 a positively oriented complex structure on a neighbourhood of S^{crit}. These are such that π is (J_0, j_0)-holomorphic near E^{crit}, and the Hessian $D^2\pi$ at any critical point is nondegenerate as a complex quadratic form. The closed two-form $\Omega \in \Omega^2(E)$ must be nondegenerate on $TE_x^{\mathrm{v}} = \ker(D\pi_x)$ for each $x \in E$, and a Kähler form for J_0 in some neighbourhood of E^{crit}. $\Theta \in \Omega^1(E)$ must satisfy $\mathrm{d}\Theta = \Omega$. We also require triviality near $\partial_{\mathrm{h}} E$, which means the existence of a map (1.2) with the same properties as for exact symplectic fibrations. For brevity, exact Lefschetz fibrations will usually be denoted by (E, π) alone, as we have already done for exact symplectic fibrations.

For any $x \in E$ there is a decomposition $TE_x = TE_x^{\mathrm{h}} \oplus TE_x^{\mathrm{v}}$ with TE_x^{h} defined as in (1.3); the horizontal part is zero at critical points, and projects isomorphically to TS_z, $z = \pi(x)$, at any other point. We say that an exact Lefschetz fibration has nonnegative curvature if $\Omega | TE_x^{\mathrm{h}} \geqslant 0$ for each x. Note that if $x \notin E^{\mathrm{crit}}$ is close to a critical point, $\Omega | TE_x^{\mathrm{h}}$ is strictly positive anyway, because of the Kählerness assumption on Ω. A standard argument based on this shows

Lemma 1.5. *If $\beta \in \Omega^2(S)$ is a sufficiently positive two-form, $\Omega + \pi^*\beta$ is a symplectic form on E.*

Symplectic parallel transport for an exact Lefschetz fibration is well defined as long as one avoids the critical fibres; indeed, if one removes those fibres, the remainder is an exact symplectic fibration over $S \setminus S^{\mathrm{crit}}$. We now take a look at the structure of the critical points. Take $z_0 \in S^{\mathrm{crit}}$ and local j_0-holomorphic coordinates $\xi : U \to S$, where $U \subset \mathbb{C}$ is a neighbourhood of the origin, such that $\xi(0) = z_0$. By assumption there is a unique critical point $x_0 \in E_{z_0}$. The holomorphic Morse Lemma says that one can find a neighbourhood of the origin $W \subset \mathbb{C}^{n+1}$ and a J_0-holomorphic chart $\Xi : W \to E$ with $\Xi(0) = x_0$, such that

$$(\xi^{-1} \circ \pi \circ \Xi)(x) = x_1^2 + \cdots + x_{n+1}^2$$

is the standard nondegenerate quadratic form on $\mathbb{C}^{n+1}$. We call (ξ, Ξ) a holomorphic Morse chart. In general, it is not possible to choose Ξ in such a way that $\Xi^*\Omega$ is the standard Kähler form on $W \subset \mathbb{C}^{n+1}$; however, one can remedy this by a suitable local deformation.

Lemma 1.6. *Let $(E, \pi, \Omega, \Theta, J_0, j_0)$ be an exact Lefschetz fibration, and x_0 a critical point of π. Then there are smooth families $\Omega^\mu \in \Omega^2(E)$, $\Theta^\mu \in \Omega^1(E)$, $0 \leqslant \mu \leqslant 1$, such that*

(i) $\Omega^0 = \Omega$, $\Theta^0 = \Theta$;
(ii) *for all μ, $\Omega^\mu = \Omega^0$ and $\Theta^\mu = \Theta^0$ outside a small neighbourhood of x_0;*
(iii) *each $(E, \pi, \Omega^\mu, \Theta^\mu, J_0, j_0)$ is an exact Lefschetz fibration;*
(iv) *there is a holomorphic Morse chart (ξ, Ξ) around x_0 such that $\Xi^*\Omega^1$, $\Xi^*\Theta^1$ agree near the origin with the standard forms $\omega_{\mathbb{C}^{n+1}} = \frac{i}{2} \sum \mathrm{d}x_k \wedge \mathrm{d}\bar{x}_k$, $\theta_{\mathbb{C}^{n+1}} = \frac{i}{4} \sum x_k \, \mathrm{d}\bar{x}_k - \bar{x}_k \, \mathrm{d}x_k$.*

The proof is based on an elementary local statement about Kähler forms.

Lemma 1.7. *Let ω be a Kähler form on the ball $B = B^{2n+2}(r)$ of radius $r > 0$ in $\mathbb{C}^{n+1}$. Then there is another Kähler form ω' which agrees with ω near ∂B, and which close to the origin is some small multiple of $\omega_{\mathbb{C}^{n+1}}$.*

Proof. The first step is to find a Kähler form ω'' which is equal to ω near ∂B, and which has constant coefficients near the origin. For this write[2] $\omega = \beta + dd^c f$, with β constant and f vanishing to second order at $x = 0$. Take a cutoff function $g \in C^\infty(\mathbb{R}^+, \mathbb{R})$ such that $g(t) = 1$ for $t \leqslant 1$ and $g(t) = 0$ for $t \geqslant 2$, and set $f_\varepsilon(x) = g(\|x\|/\varepsilon) f(x)$. A straightforward computation shows that as $\varepsilon \to 0$, the functions f_ε not only have increasingly small support, but also tend to 0 in the C^2 topology. The desired form is, for small ε,

$$\omega'' = \omega - dd^c f_\varepsilon = \beta + dd^c(f - f_\varepsilon). \tag{1.5}$$

In a second step, choose some small $t > 0$ such that the constant form $\beta' = \beta - t\omega_{\mathbb{C}^{n+1}}$ is still Kähler. Take a two-form γ on $\mathbb{C}^{n+1}$ which is of type (1,1) and nonnegative everywhere, which vanishes near the origin, and which is equal to $\omega_{\mathbb{C}^{n+1}}$ outside a compact subset; this can be obtained as $\gamma = -dd^c h(\|x\|^2)$ for a suitable convex function h. Pulling γ back by a linear map transforms it into another nonnegative (1,1)-form γ', zero near the origin and equal to β' outside a compact subset. Then $\gamma'' = \gamma' + t\omega_{\mathbb{C}^{n+1}}$ is Kähler, equals $t\omega_{\mathbb{C}^{n+1}}$ near the origin, and β outside a compact subset. That compact subset can be made arbitrarily small by retracting linearly and rescaling; the two-form obtained in that way can be plugged into ω'' locally near zero, yielding ω'. □

Proof of Lemma 1.6. Take some holomorphic Morse chart (ξ, Ξ) for x_0. Lemma 1.7 says that one can find a two-form Ω^1 on E which agrees with $\Omega = \Omega^0$ outside $\mathrm{im}(\Xi)$, such that $\Xi^*\Omega^1$ is Kähler and, near the origin, equals $c\omega_{\mathbb{C}^{n+1}}$ for some small constant $c > 0$. The obstruction to finding a $\Theta^1 \in \Omega^1(E)$ which agrees with $\Theta = \Theta^0$ outside $\mathrm{im}(\Xi)$ and satisfies $\mathrm{d}\Theta^1 = \Omega^1$ lies in $H^2(B, \partial B; \mathbb{R})$, with B a $(2n+2)$-dimensional ball; which is zero. An arbitrarily chosen Θ^1 needs to be modified to make $\Xi^*\Theta^1$ equal to $c\theta_{\mathbb{C}^{n+1}}$ near $x = 0$, but that can be done by adding the differential of some function to it. By restricting ξ, Ξ to smaller neighbourhoods, and rescaling them by $c^{-1/2}$ and c^{-1}, respectively, one achieves that $\Xi^*\Omega^1 = \omega_{\mathbb{C}^{n+1}}$ and $\Xi^*\Theta^1 = \theta_{\mathbb{C}^{n+1}}$ near the origin. Finally, Ω^μ and Θ^μ are defined by interpolating linearly between $\mu = 0$ and 1; the required properties are obvious. □

1.2. The local model

Consider $T = T^*S^n$ with its standard forms $\omega_T \in \Omega^2(T)$, $\theta_T \in \Omega^1(T)$. For concrete computations we use the coordinates $T = \{(u,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \langle u, v\rangle = 0,\ \|v\| = 1\}$. For each $\lambda > 0$, the subspace $T(\lambda)$ of cotangent vectors of length $\leqslant \lambda$ is an exact symplectic manifold. We write similarly $T(0) \subset T$ for the zero-section. The length function $\mu : T \to \mathbb{R}$, $\mu(u,v) = \|u\|$, generates a Hamiltonian circle action σ on $T \setminus T(0)$ which, after identifying $T \cong TS^n$ via the standard metric, can be described as the normalized geodesic flow on S^n. In coordinates

$$\sigma_t(u,v) = \left(\cos(t)u - \sin(t)\|u\|v, \cos(t)v + \sin(t)\frac{u}{\|u\|}\right).$$

σ_π is the antipodal involution $A(u,v) = (-u,-v)$, hence extends continuously over $T(0)$ (unlike any σ_t, $0 < t < \pi$). This can be used to define certain symplectic automorphisms of T. The construction

[2] The definition of d^c in this paper is such that $\omega_{\mathbb{C}} = -dd^c \frac{1}{4}|z|^2$. This differs from the majority convention by a negative constant.

is by now well known, but we repeat it here since precise control over the parameters will be important later on.

Lemma 1.8. *Let $R \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function which vanishes for $t \geqslant 0$ and which satisfies $R(-t) = R(t) - kt$ for small $|t|$, with some $k \in \mathbb{Z}$. Let (ϕ_t^H) be the Hamiltonian flow of $H = R(\mu)$, defined on $T \setminus T(0)$. Then $\phi_{2\pi}^H$ extends smoothly over $T(0)$ to a compactly supported symplectic automorphism ϕ of T. The function $K = 2\pi(R'(\mu)\mu - R(\mu))$ also extends smoothly over $T(0)$, and*

$$\phi^*\theta_T - \theta_T = \mathrm{d}K. \tag{1.6}$$

Proof. Two Hamiltonians H_1, H_2 which are both functions of μ always Poisson-commute, so that $\phi_t^{H_1}\phi_t^{H_2} = \phi_t^{H_1+H_2}$. Decomposing H into $H_1 = R(\mu) - (k/2)\mu$ and $H_2 = (k/2)\mu$, one gets

$$\phi_{2\pi}^H = \phi_{2\pi}^{H_1} \circ \sigma_{k\pi}.$$

We may assume that $R(-t) = R(t) - kt$ holds everywhere, since that can be achieved by modifying R for negative values only, which does not affect ϕ. Then $R(t) - (k/2)t$ is an even function, so it can be written as a smooth function of t^2. This proves that H_1 and its flow extend smoothly over $T(0)$. We know that $\sigma_{k\pi} = A^k$ extends smoothly, so the same holds for $\phi_{2\pi}^H$. Since $H(y)$ vanishes for points with $\mu(y) \geqslant 0$, ϕ is compactly supported (one can show that the compactly supported symplectic automorphisms which are obtained in this way are precisely those which are equivariant for the obvious $O(n+1)$-action). The function $R'(t)t - R(t)$ is even, so K extends smoothly over $T(0)$ for the same reason as before. A computation shows that the Hamiltonian vector field X of H satisfies $L_{2\pi X}\theta_T = \mathrm{d}K$. Since H and K are both functions of μ, ϕ_t^H preserves K, which implies (1.6). □

Clearly, if $\mathrm{supp}(R) \subset (-\infty; \lambda)$ then ϕ restricts to a symplectic automorphism of $T(\lambda)$ which is the identity near the boundary. Eq. (1.6) shows that this is an exact symplectic automorphism. In the case $k = 1$ we call these automorphisms *model Dehn twists*, and generally denote them by τ; any two of them are isotopic in $\mathrm{Symp}^e(T(\lambda))$. An explicit formula is

$$\tau(y) = \begin{cases} \sigma_{2\pi R'(\mu(y))}(y), & y \in T(\lambda) \setminus T(0), \\ A(y), & y \in T(0), \end{cases} \tag{1.7}$$

where the angle of rotation goes from $2\pi R'(0) = \pi$ to $2\pi R'(\lambda) = 0$. Note that τ maps $T(0)$ to itself, and is the antipodal map on it. If one considers $T(0)$ as an exact Lagrangian submanifold, with a function $K_{T(0)} = \text{const.}$ associated to it, then by (1.1) and (1.6)

$$K_{\tau(T(0))} = (K_{T(0)} + K_{\tau_L}|T(0)) \circ \tau_L^{-1} = K_{T(0)} - 2\pi R(0). \tag{1.8}$$

On occasion, it is useful to demand that the angle $R'(t)$ does not oscillate too much. We say that τ is *δ-wobbly* for some $0 < \delta < \frac{1}{2}$ if $R'(t) \geqslant 0$ for all $t \geqslant 0$, and $R''(t) < 0$ for all $t \geqslant 0$ such that $R'(t) \geqslant \delta$.

Lemma 1.9. *Suppose that τ is δ-wobbly. Let $F_0 = T(\lambda)_{y_0}$, $F_1 = T(\lambda)_{y_1}$ be the fibres of $T(\lambda) \to S^n$ at points y_0, y_1, whose distance in the standard metric is $\mathrm{dist}(y_0, y_1) \geqslant 2\pi\delta$. Then $\tau(F_0)$ intersects*

F_1 transversally and at a single point y, which satisfies

$$2\pi R'(\|y\|) = \mathrm{dist}(y_0, y_1).$$

In the special case where $y_1 = A(y_0)$ one has $y = y_1$; and then the tangent space of $T(\lambda)$ at y can be identified symplectically with $\mathbb{C}^n$ in such a way that the subspaces tangent to $\tau(F_0)$, $T(0)$, F_1 become, respectively, $\mathbb{R}^n$, $e^{2\pi i/3}\mathbb{R}^n$, and $e^{\pi i/3}\mathbb{R}^n$.

Proof. Consider first the case when $y_1 \neq A(y_0)$. Suppose that $y \in F_1$ is a point with $\tau^{-1}(y) \in F_0$. By identifying $T \cong TS^n$ and using the interpretation of σ as normalized geodesic flow, one sees that y must be a positive multiple of $c'(1)$, where $c : [0;1] \to S^n$ is the minimal geodesic from $c(0) = y_0$ to $c(1) = y_1$. Moreover, the angle of rotation must be $2\pi R'(\|y\|) = \|c'(1)\| = \mathrm{dist}(y_0, y_1)$. These two conditions are also sufficient. δ-wobblyness implies that $2\pi R'(t) = \mathrm{dist}(y_0, y_1)$ has exactly one solution $t > 0$, which proves that there is exactly one y. Combine the two conditions above into one, $c'(1) = 2\pi R'(\|y\|)(y/\|y\|)$. Taking the derivative, one sees that a vector $Y \in T(F_1)_y \cong T(S^n)_{y_1}$ satisfies $(T\tau)^{-1}(Y) \in T(F_0)$ iff

$$R''(\|y\|)\left\langle \frac{y}{\|y\|}, Y\right\rangle \frac{y}{\|y\|} + R'(\|y\|)\left(Y - \left\langle \frac{y}{\|y\|}, Y\right\rangle \frac{y}{\|y\|}\right) = 0.$$

We know that $R'(\|y\|) \geqslant \delta$; by δ-wobblyness this implies $R''(\|y\|) < 0$, which shows that $Y = 0$. Therefore $y \in \tau(F_0) \cap F_1$ is a transverse intersection point.

Now consider the case when $y_1 = A(y_0)$. Then $y = y_1$ clearly lies in $\tau(F_0) \cap F_1$, and because $R'(t) < \frac{1}{2}$ for all $t > 0$, there is no other intersection point. In the notation from the proof of Lemma 1.8, $\tau(F_0) = \phi_{2\pi}^{H_1}(A(F_0)) = \phi_{2\pi}^{H_1}(F_1)$, and y is a stationary point of $(\phi_t^{H_1})$. It follows that $T(\tau(F_0))_y$ is the image of $T(F_1)_y$ under the time 2π map of the linear Hamiltonian flow generated by the quadratic form $\frac{1}{2}\mathrm{Hess}(H_1)_y$. If one identifies the tangent space to T at y with $T(S^n)_y \oplus T(S^n)_y$ in such a way that the first summand is $T(F_1)_y$, then

$$\mathrm{Hess}(H_1)_y = \begin{pmatrix} R''(0)\cdot I & 0 \\ 0 & 0 \end{pmatrix}.$$

Taking some isomorphism $T(S^n)_y \cong \mathbb{R}^n$ and its complexification $T(S^n)_y \oplus T(S^n)_y \cong \mathbb{C}^n$, one finds that the tangent spaces of $\tau(F_0)$, $T(0)$, and F_1 at y correspond, respectively, to

$$(1 + 2\pi i R''(0))\mathbb{R}^n,\ i\mathbb{R}^n,\ \mathbb{R}^n \subset \mathbb{C}^n. \tag{1.9}$$

In particular, since $R''(0) < 0$ by δ-wobblyness, the intersection $\tau(F_0) \cap F_1$ is transverse. At this point we need to recall a fact from symplectic linear algebra, see, e.g. [13, p. 40]: the classification of triples of mutually transverse linear Lagrangian subspaces, up to the action of $\mathrm{Sp}(2n)$, is equivalent to the classification of nondegenerate quadratic forms on $\mathbb{R}^n$, up to $\mathrm{GL}(n, \mathbb{R})$. In particular there is a finite number of equivalence classes, and deforming a triple continuously while keeping transversality will not change its equivalence class. One can clearly deform the three subspaces (1.9) in this way to $\mathbb{R}^n$, $e^{2\pi i/3}\mathbb{R}^n$, $e^{\pi i/3}\mathbb{R}^n$, which proves the last part of the statement. □

The next result links model Dehn twists to exact Lefschetz fibrations. The connection has been known to algebraic geometers for a very long time, as attested by the terminology “Picard–Lefschetz transformations" used for model Dehn twists. But while the traditional approach ignores symplectic forms, they are of course crucial for our purpose.

Lemma 1.10. *Fix $\lambda > 0$ and $r > 0$, and let $\bar{D}(r) \subset \mathbb{C}$ be the closed disc of radius r around the origin. There is an exact Lefschetz fibration (E, π) over $\bar{D}(r)$, together with a diffeomorphism $\phi : E_r \to T(\lambda)$ which respects both the symplectic forms and the exact one-forms, such that the following holds. Denote by $\rho \in \mathrm{Symp}^e(E_r)$ the symplectic monodromy around $\partial\bar{D}(r)$, in positive sense. Then $\tau = \phi \circ \rho \circ \phi^{-1} \in \mathrm{Symp}^e(T(\lambda))$ is a model Dehn twist.*

Proof. Take $\mathbb{C}^{n+1}$ with its standard forms $\omega_{\mathbb{C}^{n+1}}$, $\theta_{\mathbb{C}^{n+1}}$ and the function $q : \mathbb{C}^{n+1} \to \mathbb{C}$, $q(x) = x_1^2 + \cdots + x_{n+1}^2$. Even though $(\mathbb{C}^{n+1}, q)$ is clearly not an exact Lefschetz fibration (its fibres are not even compact), much of what was said in the previous section carries over to it. The horizontal subspaces

$$T(\mathbb{C}^{n+1})^{\mathrm{h}}_x = \{X \in \mathbb{C}^{n+1} : \omega_{\mathbb{C}^{n+1}}(X, \ker Dq_x) = 0\} = \mathbb{C}\bar{x} \tag{1.10}$$

define a symplectic connection away from the critical point $x = 0$, so that one has parallel transport maps $q^{-1}(c(a)) \to q^{-1}(c(b))$ along paths $c : [a; b] \to \mathbb{C}^*$. Consider the family of Lagrangian spheres

$$\Sigma_z = \sqrt{z}S^n = \{(\sqrt{z}\,y_1, \ldots, \sqrt{z}\,y_{n+1}) : y \in S^n \subset \mathbb{R}^{n+1}\} \subset q^{-1}(z), \tag{1.11}$$

$z \neq 0$, which for $z \to 0$ degenerate to $\Sigma_0 = \{0\} \subset q^{-1}(0)$. Write Σ^* for the union of all Σ_z, $z \neq 0$, and $\Sigma = \Sigma^* \cup \Sigma_0$. One computes that

$$\theta_{\mathbb{C}^{n+1}} \,|\, \Sigma^* = q^* d^c(-\tfrac{1}{4}|z|). \tag{1.12}$$

Actually, the precise formula does not matter much for the moment. All we need is that $\omega_{\mathbb{C}^{n+1}} \,|\, \Sigma^*$ is the pullback by q of some two-form on $\mathbb{C}^*$, since that implies that parallel transport in any direction in $\mathbb{C}^*$ takes the Σ_z into each other; cf. Lemma 1.3. The next observation is that if one removes Σ then parallel transport can be extended even to the singular fibre, so that for any path c in $\mathbb{C}$ one has a canonical symplectic isomorphism

$$q^{-1}(c(a)) \setminus \Sigma_{c(a)} \to q^{-1}(c(b)) \setminus \Sigma_{c(b)}.$$

Since taking out Σ_0 removes the critical point, the only possible problem is that a point in $\mathbb{C}^{n+1} \setminus \Sigma$ might move in horizontal direction and converge to some point of Σ, which means that the flow of some horizontal vector field would not be defined for all time. However, that cannot happen since there is a function, $h(x) = \|x\|^4 - |q(x)|^2$, which satisfies $\mathrm{d}h_x(\mathbb{C}\bar{x}) = 0$, hence is constant horizontally, and with $h^{-1}(0) = \Sigma$. Therefore one can use parallel transport in radial direction to trivialize $q : \mathbb{C}^{n+1} \setminus \Sigma \to \mathbb{C}$ symplectically. In particular, if $\tilde{\rho}_s : q^{-1}(s) \to q^{-1}(s)$ is the monodromy along the circle of radius $s > 0$ around the origin, $\tilde{\rho}_s \,|\, (q^{-1}(s) \setminus \Sigma_s)$ will be isotopic to the identity in the group of all symplectic automorphisms of $q^{-1}(s) \setminus \Sigma_s$. We have mentioned all this mainly to motivate the subsequent proof, which is more computational.

Consider the map

$$\Phi : \mathbb{C}^{n+1} \setminus \Sigma \to \mathbb{C} \times (T \setminus T(0)),$$

$$\Phi(x) = (q(x), \sigma_{\alpha/2}(-\mathrm{im}(\hat{x})\,\|\mathrm{re}(\hat{x})\|, \mathrm{re}(\hat{x})\|\mathrm{re}(\hat{x})\|^{-1})), \tag{1.13}$$

where $se^{i\alpha} = q(x)$ are polar coordinates on the base, and $\hat{x} = e^{-i\alpha/2}x$. We claim that this is a diffeomorphism fibered over $\mathbb{C}$. First of all, because of the use of polar coordinates, it is not obvious that Φ is well defined and smooth at $q^{-1}(0) \setminus \Sigma_0$. To dispel any doubts about that one writes, after some manipulations,

$$\Phi(x) = (q(x), -\tfrac{1}{2}\,\mathrm{im}(x)\beta(x) - \tfrac{1}{2}\,\mathrm{im}(\overline{q(x)}x)\beta(x)^{-1},$$
$$h(x)^{-1/2}\,\mathrm{re}(x)\beta(x) - h(x)^{-1/2}\mathrm{re}(\overline{q(x)}x)\beta(x)^{-1}),$$

where $\beta(x) = (\|x\|^2 + h(x)^{1/2})^{1/2}$. The fact that Φ is a diffeomorphism on each fibre is easy to see, either directly or by using the symplectic forms and (1.14). Moreover, if one restricts Φ to a fibre over $s > 0$, it extends to a diffeomorphism $\phi_s : q^{-1}(s) \to T$. Such an extension does not exist for other fibres, due to the noncontinuity of σ at $T(0)$. A computation yields

$$(\Phi^{-1})^*\theta_{\mathbb{C}^{n+1}} = \theta_T - \tilde{R}_s(\mu)\,d\alpha, \qquad \tilde{R}_s(t) = \tfrac{1}{2}t - \tfrac{1}{2}(t^2 + s^2/4)^{1/2}. \tag{1.14}$$

This implies that the restriction of Φ to any fibre is symplectic, and actually maps the respective one-forms into each other. Of course, the same will then be true for the continuous extensions ϕ_s, $s > 0$. In fact (1.14) shows even more: if one restricts to any ray $\alpha = \text{const.}$ in the basis then $\Phi^*\omega_T = \omega_{\mathbb{C}^{n+1}}$, which means that Φ trivializes the symplectic parallel transport on $(\mathbb{C}^{n+1} \setminus \Sigma, q)$ in radial directions, in accordance with the strategy which we set out before. Consider $\tilde{\tau}_s = \phi_s \circ \tilde{\rho}_s \circ \phi_s^{-1} : T \to T$, where $\tilde{\rho}_s$ is the monodromy map introduced above. From (1.14) it follows that $\tilde{\tau}_s$ restricted to $T \setminus T(0)$ is the time 2π map of the Hamiltonian $\tilde{H}_s = \tilde{R}_s(\mu)$. Since $\tilde{R}_s(-t) = \tilde{R}_s(t) - t$, this is quite close to the case $k = 1$ of Lemma 1.8. The difference is that $\tilde{R}_s(t)$ does not vanish for $t \gg 0$. Instead, it decays as follows:

$$0 > \tilde{R}_s(t) \geqslant -\frac{1}{16}s^2t^{-1}, \quad 0 < \frac{d}{dt}\tilde{R}_s(t) \leqslant \frac{1}{16}s^2t^{-2}. \tag{1.15}$$

This is good enough to imply that $\tilde{\tau}_s$ is asymptotic to the identity at infinity, for each $s > 0$. It remains to tweak the given data slightly, so as to produce a honest exact Lefschetz fibration, whose monodromy is an actual model Dehn twist.

Fix $\lambda > 0$, $r > 0$. Choose a cutoff function $g \in C^\infty(\mathbb{R}^+, \mathbb{R})$ such that $g'(t) \geqslant 0$ everywhere, $g(t) = 0$ for small t, and $g(t) = 1$ if t is close to λ. We claim that there is a unique $\gamma \in \Omega^1(\mathbb{C}^{n+1})$ with, again in polar coordinates on the base,

$$(\Phi^{-1})^*\gamma = g(\mu)\tilde{R}_s(\mu)\,d\alpha. \tag{1.16}$$

Since $\tilde{R}_0(t) = 0$, the function $\tilde{R}_s(t)/s$ extends smoothly to $s = 0$, $t \neq 0$. Therefore the right-hand side of (1.16) can be written as $g(\mu)(\tilde{R}_s(\mu)/s)s\,d\alpha$, which means that γ is smooth at least on $\mathbb{C}^{n+1} \setminus \Sigma$. Now $\mu(\Phi(x)) = \frac{1}{2}h(x)^{1/2}$; since $\Sigma = h^{-1}(0)$ and $g(t) = 0$ for small t, one sees that $\Phi^*(g(\mu)\tilde{R}_s(\mu)\,d\alpha)$ vanishes near Σ, so that γ extends by zero over Σ. Set

$$E = \Phi^{-1}(\bar{D}(r) \times (T(\lambda) \setminus T(0))) \cup (\Sigma \cap q^{-1}(\bar{D}(r))),$$
$$\pi = q|E : E \to \bar{D}(r),$$
$$\Theta = (\theta_{\mathbb{C}^{n+1}} + \gamma)\,|\,E, \quad \Omega = d\Theta,$$
$$\phi = \phi_r|E_r : E_r \to T(\lambda). \tag{1.17}$$

$E \subset \mathbb{C}^{n+1}$ is cut out by the inequalities $h(x) \leqslant 4\lambda^2$, $|q(x)| \leqslant r$. This makes it easy to show that it is a compact manifold with corners, whose boundary faces are $\partial_{\mathrm{v}}E = \pi^{-1}(\partial\bar{D}(r))$ and $\partial_{\mathrm{h}}E = \{h(x) = 4\lambda^2\} = \{\mu(\Phi(x)) = \lambda\} = \Phi^{-1}(\bar{D}(r) \times \partial T(\lambda))$. Because γ vanishes when restricted to any fibre, $\Omega|TE^{\mathrm{v}}_x = \omega_{\mathbb{C}^{n+1}}|\ker Dq_x$ is nondegenerate for all x. In

$$\Theta \,|\, (E \setminus \Sigma) = \Phi^*(\theta_T + (g(\mu) - 1)\tilde{R}_s(\mu)\,\mathrm{d}\alpha) \tag{1.18}$$

the second term on the right-hand side vanishes close to $\bar{D}(s) \times \partial T(\lambda)$, so that Φ provides a trivialization near $\partial_{\mathrm{h}}E$ in the sense introduced in the previous section. Moreover, since γ vanishes near the critical point $x = 0$, equipping E with the standard complex structure J_0 near that point, and $\bar{D}(r)$ with the standard complex structure j_0, turns (E, π) into an exact Lefschetz fibration. As for the statement about the monodromy ρ, one can repeat the argument above, using (1.18) instead of (1.14). This shows that $\tau = \phi \circ \rho \circ \phi^{-1}$, when restricted to $T(\lambda) \setminus T(0)$, is the time 2π map of $R_r(\mu)$, where

$$R_r(t) = (1 - g(t))\tilde{R}_r(t), \tag{1.19}$$

by definition, this is a model Dehn twist. □

Let (M, ω, θ) be an exact symplectic manifold. A *framed*[3] *exact Lagrangian sphere* is an exact Lagrangian submanifold $L \subset M$ together with an equivalence class $[f]$ of diffeomorphisms $f : S^n \to L$. Here f_1, f_2 are equivalent iff $f_2^{-1} f_1$ can be deformed inside $\mathrm{Diff}(S^n)$ to an element of $O(n+1)$. To any such $(L, [f])$ one associates a Dehn twist $\tau_{(L,[f])} \in \mathrm{Symp}^{\mathrm{e}}(M)$ as follows. Choose a representative f and extend it to a symplectic embedding $\imath : T(\lambda) \to M$ for some $\lambda > 0$. Take a model Dehn twist τ which is supported in the interior of $T(\lambda)$, and define

$$\tau_{(L,[f])} = \begin{cases} \imath \circ \tau \circ \imath^{-1} & \text{on } \mathrm{im}(\imath), \\ \mathrm{id} & \text{elsewhere.} \end{cases}$$

The exactness of $\tau_{(L,[f])}$ follows from that of L; moreover, the analogue of (1.8) holds. It is not difficult to show that the isotopy class $[\tau_{(L,[f])}] \in \pi_0(\mathrm{Symp}^{\mathrm{e}}(M))$ is independent of the choices made in the definition. In contrast, it is unknown whether a change of the framing $[f]$ can affect $[\tau_{(L,[f])}]$; if the answer is negative the notion of framing could be dropped altogether, but while the question is open one cannot do without it. Still, for the sake of brevity we will often omit framings from the notation and write τ_L instead of $\tau_{(L,[f])}$. We will say that τ_L is δ-wobbly if the local model it is constructed out of has this property.

Proposition 1.11. *Let $(L, [f])$ be a framed exact Lagrangian sphere in M. Fix some $r > 0$. There is an exact Lefschetz fibration (E^L, π^L) over $\bar{D}(r)$ together with an isomorphism $\phi^L : E^L_r \to M$ of exact symplectic manifolds, such that if ρ^L is the symplectic monodromy around $\partial\bar{D}(r)$, then $\tau_L = \phi^L \circ \rho^L \circ (\phi^L)^{-1}$ is a Dehn twist along $(L, [f])$.*

We will prove this under the assumption that there is an embedding $\imath : T(\lambda) \to M$ as before, with $\imath^*\theta = \theta_T$. This is not really a restriction, since one can always satisfy it by adding the derivative of a function to θ, which does not change M up to exact symplectic isomorphism. On the other hand,

[3] This has little or nothing to do with the usual topological notion of framed manifold.

it allows us to make the statement slightly sharper: ϕ^L will map the one-forms on E^L_r and M into each other, and the Dehn twist obtained from the monodromy of (E^L, π^L) will be one constructed using the given embedding $\imath$.

Proof. Take (E, π) and ϕ from Lemma 1.10, with the given r and λ. We will construct (E^L, π^L) by attaching a trivial piece to (E, π). By construction, there is a neighbourhood $N \subset E$ of $\partial_{\mathrm{h}} E$, a neighbourhood V of $\partial T(\lambda)$ in $T(\lambda)$, and a diffeomorphism $\Phi : N \to \bar{D}(r) \times V$ fibered over $\bar{D}(r)$, such that $\Theta | N = \Phi^* \theta_T$. Moreover, Φ agrees with ϕ on $N \cap E_r$. Set

$$E^L = E \cup_{\sim} \bar{D}(r) \times (M \setminus \imath(T(\lambda) \setminus V)),$$

where $\sim$ identifies N with $\bar{D}(r) \times \imath(V)$ through $(\mathrm{id} \times \imath) \circ \Phi$. One similarly defines π^L from π and the projection $\bar{D}(r) \times M \to \bar{D}(r)$. The forms Ω^L, Θ^L on E^L come from the corresponding ones on E and the pullbacks of ω, θ on the trivial part. ϕ^L is constructed from $\imath \circ \phi$ and the identity map. The complex structure near the critical point and critical value are inherited from E. All properties stated above are obvious from the construction and the definition of Dehn twists. $\square$

We call the (E^L, π^L) *standard fibrations*. For future reference, we will now state certain properties which these fibrations inherit from the local model (E, π), and which depend on the details of its construction.

Lemma 1.12. *Any standard fibration (E^L, π^L) has the following properties.*

(i) *There is a closed subset $\Sigma^L \subset E^L$ such that*

$$\Sigma^L_z = \Sigma^L \cap E^L_z = \begin{cases} \text{is an embedded } n\text{-sphere} & \text{if } z \neq 0, \\ \text{is the unique critical point } x_0 \in E^L_0 & \text{if } z = 0. \end{cases}$$

In fact $(\Sigma^L)^* = \Sigma^L \setminus \Sigma^L_0$ is a smooth n-sphere bundle over $\bar{D}(r) \setminus \{0\}$, and satisfies

$$\Theta^L \,|\, (\Sigma^L)^* = (\pi^L)^* d^c(-\tfrac{1}{4}|z|). \tag{1.20}$$

As in the discussion following (1.12), this implies that each $\Sigma^L_z \subset E^L_z$, $z \neq 0$, is an exact Lagrangian submanifold, and that symplectic parallel transport within $\bar{D}(r) \setminus \{0\}$ carries these spheres into each other. Moreover, $\phi^L(\Sigma^L_r) = L$.

(ii) *There are holomorphic Morse charts (ξ, Ξ) around the unique critical point $x_0 \in E^L_0$, such that ξ is the inclusion $U \hookrightarrow \bar{D}(r)$ of some neighbourhood $U \subset \mathbb{C}$ of the origin; $\Xi^* \Theta^L = \theta_{\mathbb{C}^{n+1}}$ and $\Xi^* \Omega^L = \omega_{\mathbb{C}^{n+1}}$; and $\Xi^{-1}(\Sigma^L_z) = \Sigma_z$ for all sufficiently small $z \in \mathbb{C}$.*

(iii) *(E^L, π^L) has nonnegative curvature.*

(iv) *$\tau_L = \phi^L \circ \rho^L \circ (\phi^L)^{-1}$ is the Dehn twist defined using $\imath$ and the function R_r from (1.19); in particular $R_r(0) = -r/4$. By making r smaller while keeping all other choices in the construction fixed, one can achieve that τ_L is δ-wobbly for an arbitrary δ.*

Proof. (i) From (1.17) one sees that $\Sigma \cap q^{-1}(\bar{D}(r)) \subset E$, and that $\Theta = \theta_{\mathbb{C}^{n+1}}$ in a neighbourhood of that subset. Σ^L is defined to be the image of this in E^L, so that (1.20) is a consequence of (1.12).

(ii) The definition of ξ, Ξ is obvious, and the claim about $\Xi^* \Theta^L$ follows from the fact that in (1.17) $\Theta = \theta_{\mathbb{C}^{n+1}}$ near the critical point.

(iii) One can compute the curvature of (E, π) from (1.18); it turns out to be

$$\Phi^* \left((g(\mu) - 1) \frac{\partial}{\partial s} \tilde{R}_s(\mu) \right) \mathrm{d}s \wedge \mathrm{d}\alpha,$$

which is $\geqslant 0$ everywhere. This implies the same property for (E^L, π^L).

(iv) The nontrivial statement is δ-wobblyness, which requires that we take another look at the function R_r. There is a $t_0 > 0$ such that $g(t) = 0$ for $t \in [0; t_0]$, and in that interval $R_r(t) = \tilde{R}_r(t)$, so that $(\partial/\partial t) R_r(t) > 0$, $(\partial^2/\partial t^2) R_r(t) < 0$. For $t \in [t_0; \lambda]$ one estimates using (1.15) that

$$0 \leqslant \frac{\partial}{\partial t} R_r(t) = -g'(t)\tilde{R}_r(t) + (1 - g(t)) \frac{\partial}{\partial t} \tilde{R}_r(t) \leqslant \frac{r^2}{16} (\|g'\| t_0^{-1} + t_0^{-2}).$$

By choosing r small while keeping g and hence t_0 fixed, one can make this $< \delta$ for an arbitrary δ. □

1.3. Vanishing cycles

We have seen that all Dehn twists can be realized as monodromy maps (of standard fibrations). Conversely, the geometry of any exact Lefschetz fibration can be understood in terms in Dehn twists. This is again well known on a topological level, and our exposition repeats the classical arguments while paying more attention to symplectic forms. The results will not be used again in this paper, but they are important in applications.

Let (E, π) be an exact Lefschetz fibration over S. Take $z_0 \in S^{\mathrm{crit}}$, and the unique critical point $x_0 \in E_{z_0}$. Let $c : [a; b] \to S$ be a smooth embedded path with $c(b) = z_0$, $c^{-1}(S^{\mathrm{crit}}) = \{b\}$; and let $\rho_{c|[s,s']} : E_{c(s)} \to E_{c(s')}$ be the parallel transport maps along it, which is defined for all $s \leqslant s' < b$. Following a suggestion of Donaldson, we define

$$B_c = \left\{ x \in E_{c(s)},\ a \leqslant s < b : \lim_{s' \to b} \rho_{c|[s;s']}(x) = x_0 \right\} \cup \{x_0\} \subset E. \tag{1.21}$$

Lemma 1.13. *B_c is an embedded closed $(n+1)$-ball, with $\partial B_c = B_c \cap E_{c(a)}$. The function $p = c^{-1} \circ \pi : B_c \to [a; b]$ has x_0 as its unique critical point, which is a nondegenerate local maximum. Moreover, $\Omega | B_c = 0$.*

Proof. Put a symplectic form $\Omega + \pi^*\beta$ on E as in Lemma 1.5. Choose an oriented embedding $\tilde{c} : (a - \varepsilon; b + \varepsilon) \times (-\varepsilon; \varepsilon) \to S$ such that $\tilde{c}(0, t) = c(t)$. Let h be the function defined on $\mathrm{im}(\tilde{c})$ with $h(\tilde{c}(s, t)) = -t$. The Hamiltonian vector field X of $H = h \circ \pi$ has the following properties:

(i) it is horizontal everywhere, $X_x \in TE^{\mathrm{h}}_x$;
(ii) for each $x \neq x_0$, $D\pi(X)_x$ is a positive multiple of $\partial\tilde{c}/\partial s$;
(iii) x_0 is a hyperbolic stationary point of X, with $n+1$ positive and negative eigenvalues.

Properties (i) and (ii) are straightforward; (iii) can be seen by looking at X in a holomorphic Morse chart, where it is $J_0 \nabla H$. Let $\tilde{B} \subset E$ be the stable manifold of x_0. It is an open $(n+1)$-ball, lies in $\pi^{-1}\tilde{c}((a-\varepsilon;b]\times\{0\})$, and the projection

$$\tilde{p}=\tilde{c}^{-1}\pi:\tilde{B}\to(a-\varepsilon;b]\times\{0\}$$

is a proper map. The tangent space of $\tilde{B}$ at x_0 is the negative eigenspace of DX_{x_0}; one shows easily that $D^2\tilde{p}|T\tilde{B}_{x_0}$ differs from $DX|T\tilde{B}_{x_0}$ only by a positive constant, which means that x_0 is a nondegenerate maximum of $\tilde{p}$. There are no other critical points, because elsewhere $\mathrm{d}\tilde{p}(X)_x>0$ by (ii). Finally, since the flow X is symplectic and contracts the tangent spaces of $\tilde{B}$, these must be Lagrangian subspaces, so $(\Omega+\pi^*\beta)|\tilde{B}=\Omega|\tilde{B}=0$.

We claim that $B_c=\tilde{B}\cap\pi^{-1}(\mathrm{im}(c))$, which implies all desired properties of B_c. Let Y be the vector field on $\pi^{-1}(\mathrm{im}(\tilde{c}))\setminus\{x_0\}$ which is horizontal and satisfies $D\pi(Y)=\partial\tilde{c}/\partial s$. From (i) and (ii) above one sees that $Y=gX$ for some function g, bounded from below by a positive constant, and which goes to ∞ as one approaches x_0. Therefore the orbits of X and Y coincide, except of course for $\{x_0\}$. Since $Y|\pi^{-1}(\mathrm{im}(c))$ defines parallel transport along c, the claim follows. □

Lemma 1.14. *$V_c=\partial B_c$ is an exact Lagrangian sphere in $E_{c(a)}$, and comes with a canonical framing.*

Proof. Since B_c is a ball and $\Theta|B_c$ is closed, there is a unique $K\in C^\infty(B_c,\mathbb{R})$ with $K(x_0)=0$ and $\mathrm{d}K=\Theta|B_c$. The restriction $K_{V_c}=K|V_c$ makes V_c into an exact Lagrangian submanifold. It remains to explain the framing. In any chart on B_c around x_0, the level sets $p^{-1}(s')$ of the function from Lemma 1.13, for s' close to b, will be strictly convex hypersurfaces, so that one can map them to S^n by radial projection. On the other hand, $V_c=p^{-1}(a)$ can be identified with $p^{-1}(s')$ by the gradient flow with respect to some metric. Combining these two maps gives a diffeomorphism $V_c\to S^n$, which is unique up to isotopy and action of $O(n+1)$; its inverse is our framing. □

The framed exact Lagrangian sphere $V_c\subset E_{c(a)}$ is called the *vanishing cycle* associated to c. It exists more generally for any path $c:[a;b]\to S$ which is smooth, not necessarily embedded, but still satisfies

$$c^{-1}(S^{\mathrm{crit}})=\{b\},\qquad c'(b)\neq 0. \tag{1.22}$$

To construct it in this situation, one first extends c to a map $(a-\varepsilon;b+\varepsilon)\times(-\varepsilon;\varepsilon)\to S$ which is a local oriented diffeomorphism at $(b,0)$, pulls back (E,π) by that map, and then applies Lemma 1.13 to the pullback exact Lefschetz fibration. Note also that deforming c smoothly, rel endpoints, within the class (1.22) yields an exact Lagrangian isotopy of the corresponding vanishing cycles, which is compatible with their framings.

Proposition 1.15. *Let c be a path satisfying (1.22), and l the loop in $S\setminus S^{\mathrm{crit}}$ obtained by "doubling" c, as in Fig. 1. Then the monodromy around l is isotopic to the Dehn twist along the vanishing cycle V_c:*

$$[\rho_l]=[\tau_{V_c}]\in\pi_0(\mathrm{Symp}^{\mathrm{e}}(E_{c(a)})). \tag{1.23}$$

Proof. We start with a rather special case. Let (E,π) be an exact Lefschetz fibration with base $\bar{D}(r)$ for some $r>0$, and which has exactly one critical point $x_0\in E_0$. Write $M=E_r$. In addition, x_0 should admit a holomorphic Morse chart (ξ,Ξ) where Ξ is defined on $W=\{x\in\mathbb{C}^{n+1}:\|x\|\leqslant 2\sqrt{r},\ |q(x)|\leqslant r\}$, where $\xi=\mathrm{id}_{\bar{D}(r)}$, and such that $\Xi^*\Omega$, $\Xi^*\Theta$ are standard. This means that the symplectic geometry of

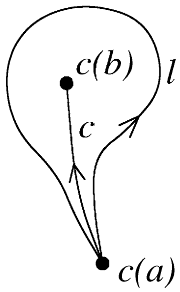

Fig. 1.

(E,π) equals that of the local model $(\mathbb{C}^{n+1},q)$ discussed in Lemma 1.10, at least in a suitable neighbourhood of x_0. In particular, each fibre E_z, $z\neq 0$, contains an exact Lagrangian sphere $\Sigma_z^E=\Xi(\Sigma_z)$, degenerating to $\Sigma_0^E=\{x_0\}$. Write $L=\Sigma_r^E\subset M$; this inherits an obvious framing from $\Sigma_r=\sqrt{r}S^n$. We claim that the monodromy around $\partial\bar{D}(r)$, denoted by $\rho\in\mathrm{Symp}^{\mathrm{e}}(M)$, is isotopic to τ_L.

From (1.2) one deduces that parallel transport in $\bar{D}(r)\setminus\{0\}$ carries the Σ_z^E into each other. Moreover, by the same argument as in Lemma 1.10, if one removes $\Sigma^E=\bigcup_z\Sigma_z^E$ from E, parallel transport can be extended over the singular fibre. Using parallel transport in radial directions one constructs a trivialization $\Phi^E:E\setminus\Sigma^E\to\bar{D}(r)\times(M\setminus L)$ which is the identity on the fibre over r; this is such that, in radial coordinates $z=s\mathrm{e}^{\mathrm{i}\alpha}$ on the basis, $((\Phi^E)^{-1})^*\Theta=\theta-R^E\wedge\mathrm{d}\alpha+\mathrm{d}S^E$ for some functions $R^E=R^E(s,\alpha,x)$ and $S^E=S^E(s,\alpha,x)$ which vanish for x close to ∂M. This implies that ρ restricted to $M\setminus L$ is the time-2π map of the flow generated by the time-dependent Hamiltonian $(\alpha,x)\mapsto R^E(r,\alpha,x)$. Clearly, $[\rho]\in\pi_0(\mathrm{Symp}^{\mathrm{e}}(M))$ depends only on the behaviour of this function in an arbitrarily small neighbourhood of L, or equivalently on (E,π) and Ω close to Σ^E; since $\Sigma^E\subset\Xi(W)$ by definition, this can be determined from the local model $(\mathbb{C}^{n+1},q)$. It remains to spell out the computation.

Define an embedding $\imath:T(\lambda)\to M$, for some $\lambda>0$, by combining $\Xi_r=\Xi|(W\cap q^{-1}(r))\to M$ with the inverse of the isomorphism $\phi_r:q^{-1}(r)\to T$ from the proof of Lemma 1.10. This satisfies $\imath^*\theta=\theta_T$, $\imath^*\omega=\omega_T$, and $\imath(T(0))=L$. Because both Φ^E and the trivialization Φ from (1.13) are defined by radial parallel transport, there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{C}^{n+1}\setminus\Sigma & \xrightarrow{\ \Phi\ } & \mathbb{C}\times(T\setminus T(0)) \\
\big\uparrow & & \big\uparrow \\
W\setminus\Sigma & & \bar{D}(r)\times(T(\lambda)\setminus T(0)) \\
{\scriptstyle \Xi\,|\,W\setminus\Sigma}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\times(\imath\,|\,T(\lambda)\setminus T(0))} \\
E\setminus\Sigma^E & \xrightarrow{\ \Phi^E\ } & \bar{D}(r)\times(M\setminus L).
\end{array}$$

From this, $\Xi^*\Omega=\omega_{\mathbb{C}^{n+1}}$ and (1.4) it follows that

$$\begin{aligned}(\mathrm{id}\times\imath)^*(\theta_T-R^E\mathrm{d}\alpha+\mathrm{d}S^E)&=(\mathrm{id}\times\imath)^*((\Phi^E)^{-1})^*\Theta\\ &=(\Phi^{-1})^*\theta_{\mathbb{C}^{n+1}}=\theta_T-\tilde{R}_s(\mu)\wedge\mathrm{d}\alpha,\end{aligned}$$

and in particular that $R^E(r, \alpha, \iota(y)) = \tilde{R}_r(\mu(y))$ for all α. In view of the discussion above, and the fact that $\tilde{R}_r(-t) = \tilde{R}_r(t) - t$, this implies the desired equality $[\rho] = [\tau_L]$. Now consider the path $c : [0; r] \to \bar{D}(r)$, $c(s) = r - s$. Because symplectic parallel transport takes the Σ_z^E into each other, the ball B_c from Lemma 1.13 must be the union of Σ_z^E for all $z \in [0; r]$, and the vanishing cycle is $V_c = \Sigma_r^E = L \subset M$, with the same framing as before. Therefore, what we have done up to now proves (1.23) for this special class of exact Lefschetz fibrations (E, π) and the particular path c.

More generally, let (E, π) be a exact Lefschetz fibration with arbitrary base S, and $x_0 \in E_{z_0}$ a critical point which admits a holomorphic Morse chart (ξ, Ξ) such that $\Xi^*\Omega$, $\Xi^*\Theta$ are standard. By restricting to a suitably small disc around z_0 in the base, and making the domains of ξ, Ξ smaller, one can arrive at the situation considered before. This means that (1.23) is true at least for one (short) path c with endpoint z_0. But from that it follows easily for all other paths with the same endpoint.

It remains to remove the assumption concerning Ξ. Let (E, π) be an arbitrary exact Lefschetz fibration, c a path as in (1.22), and l a corresponding loop. With respect to the critical point in $E_{c(b)}$, take smooth families Ω^μ, Θ^μ as in Lemma 1.6. These can be chosen such that in a neighbourhood of $E_{c(a)}$, $\Omega^\mu = \Omega$ and $\Theta^\mu = \Theta$ for all μ. The corresponding vanishing cycles $V_c^\mu \subset E_{c(a)}$ form a smooth isotopy of framed exact Lagrangian spheres, and similarly there is a smooth family of monodromies ρ_l^μ. For $\mu = 1$ our previous assumption about holomorphic Morse charts is satisfied, and hence $[\rho_l] = [\rho_l^0] = [\rho_l^1] = [\tau_{V_c^1}] = [\tau_{V_c^0}] = [\tau_{V_c}] \in \pi_0(\mathrm{Symp}^e(E_{c(a)}))$. $\square$

2. Pseudo-holomorphic sections

Before, we have considered exact Lefschetz fibrations as geometric objects in the sense of elementary symplectic geometry; now we will apply the theory of pseudo-holomorphic curves to them. By today's standards, the necessary analysis is rather unsophisticated. The basic Gromov-type invariant which will be introduced first uses only the most familiar techniques, on the level of the book [15]. Of course ours is a Lagrangian boundary-value problem, but the only part of the analysis which specifically concerns the boundary is bubbling off of holomorphic discs, which was addressed by Floer [6] and Oh [16] (some more recent expositions are [10, Section A.4.3] and [7]). Later, when considering relative invariants, we will use essentially the same analysis as in [24,19,30], even though the geometric setting is rather different. The remaining material (Sections 2.2, 2.3, and 2.5) is more specifically designed for application to the exact sequence, and has a greater claim to originality.

2.1. A simple invariant

Let (E^{2n+2}, π) be an exact Lefschetz fibration over S. A *Lagrangian boundary condition* is an $(n+1)$-dimensional submanifold $Q \subset E|\partial S$ which is disjoint from $\partial_h E$ and such that $\pi|Q : Q \to \partial S$ is a submersion, together with $\kappa_Q \in \Omega^1(\partial S)$ and $K_Q \in C^\infty(Q, \mathbb{R})$, satisfying $\Theta|Q = \pi^*\kappa_Q + \mathrm{d}K_Q$. This implies that each $Q_z = Q \cap E_z$, $z \in \partial S$, is an exact Lagrangian submanifold of E_z, with function $K_Q|Q_z$ (one could consider a more general situation in which the condition on $\pi|Q$ is dropped, so that the Q_z could have singularities; but we will not do that). From Lemma 1.3 one sees that parallel transport along ∂S carries the Q_z into each other, in the ordinary sense of the word; this holds in the "exact" sense, that is to say as exact Lagrangian submanifolds, iff $\kappa_Q = 0$. Note also that if one equips E with a symplectic form $\Omega + \pi^*\beta$ as in Lemma 1.5, Q itself becomes a Lagrangian submanifold. The aim of this section is to introduce, in the case where S is compact with $\partial S \neq \emptyset$, a Gromov-type invariant

$$\Phi_1(E, \pi, Q) \in H_*(Q_\zeta; \mathbb{Z}/2),$$

where ζ is some point of ∂S. In a nutshell, this is the cycle represented by the values at ζ of pseudo-holomorphic sections of E with boundary in Q.

To begin, we remind the reader of a class of almost complex structures suitable for exact symplectic manifolds. The Liouville vector field N on such a manifold, $i_N\omega = \theta$, defines a collar $\mathbb{R}^- \times \partial M \hookrightarrow M$. Let σ be the function on a neighbourhood of ∂M whose composition with $\mathbb{R}^- \times \partial M \hookrightarrow M$ is projection to the $\mathbb{R}^-$ factor. An ω-compatible almost complex J is called convex near the boundary if $\theta \circ J = \mathrm{d}(\mathrm{e}^\sigma)$ near ∂M. This implies $\mathrm{d}(\mathrm{d}(\mathrm{e}^\sigma) \circ J) = -\omega$, which serves to control the behaviour of J-holomorphic curves. It is well known that the space of these J is contractible (in particular, nonempty).

Given an exact Lefschetz fibration (E, π), one can consider the Liouville vector field on each fibre, and this gives rise to a function σ on a neighbourhood of $\partial_h E$ in E. Choose a complex structure j on the base S (whenever we do that, now or later, j is assumed to be positively oriented and equal to j_0 in some neighbourhood of S^{crit}). An almost complex structure J on E is called *compatible relative to* j if

(i) $J = J_0$ in a neighbourhood of E^{crit};
(ii) $D\pi \circ J = j \circ D\pi$;
(iii) $\Omega(\cdot, J\cdot)|TE_x^{\mathrm{v}}$ is symmetric and positive definite for any $x \in E$;
(iv) in a neighbourhood of $\partial_h E$, $J(TE^{\mathrm{h}}) = TE^{\mathrm{h}}$ and $\Theta \circ J = \mathrm{d}(\mathrm{e}^\sigma)$.

To see more concretely the meaning of these conditions, take $x \notin E^{\mathrm{crit}}$ and split $TE_x = TE_x^{\mathrm{h}} \oplus TE_x^{\mathrm{v}} \cong TS_z \oplus TE_x^{\mathrm{v}}$, $z = \pi(x)$. One can then write

$$J_x = \begin{pmatrix} j_z & 0 \\ J_x^{\mathrm{vh}} & J_x^{\mathrm{vv}} \end{pmatrix}, \tag{2.1}$$

where $J_x^{\mathrm{vv}} \in \mathrm{End}(TE_x^{\mathrm{v}})$ is a complex structure compatible with $\Omega|TE_x^{\mathrm{v}}$, and $J_x^{\mathrm{vh}} \in \mathrm{Hom}(TS_z, TE_x^{\mathrm{v}})$ is $\mathbb{C}$-antilinear with respect to j and J^{vv}. This is a reformulation of (ii), (iii). An immediate consequence is the following result, which sharpens Lemma 1.5.

Lemma 2.1. *Let J be compatible relative to j. Then for any sufficiently positive $\beta \in \Omega^2(S)$, $\Omega + \pi^*\beta$ tames J.*

Because of the lack of antisymmetry in (2.1), J will not be compatible with $\Omega + \pi^*\beta$ in the ordinary sense of the word, unless $J^{\mathrm{vh}} = 0$; we will return to this more restricted class of almost complex structures in the next section. Continuing with the analysis of the conditions above, suppose now that $x \in E$ is sufficiently close to $\partial_h E$. More precisely, we require that $\sigma(x)$ is defined and that (iv) applies. There is then a further splitting

$$TE_x^{\mathrm{v}} \cong \mathbb{R}N \oplus \mathbb{R}R \oplus (\ker \Theta \cap \ker \mathrm{d}\sigma \cap TE_x^{\mathrm{v}}), \tag{2.2}$$

where N is the Liouville vector field and R is the Hamiltonian vector field of $e^{\sigma}|E_z$. The two parts of (iv) say that $J_x^{\mathrm{vh}} = 0$ and that with respect to (2.2),

$$J_x^{\mathrm{vv}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & * \end{pmatrix}. \tag{2.3}$$

The pointwise analysis which we have just carried out can be recast in terms of sections of fibre bundles, and one sees then that the space $\mathscr{J}(E,\pi,j)$ of almost complex structures which are compatible relative to j is contractible.

Suppose now that we have a Lagrangian boundary condition Q. The theory of pseudo-holomorphic sections with boundary in Q fits into a familiar framework of infinite-dimensional manifolds and maps. We will now review this, on a formal level, that is to say using C^{∞} spaces and without assuming that S is compact. In that sense, $\mathscr{J}(E,\pi,j)$ is an infinite-dimensional manifold; its tangent space at J consists of sections $Y \in C^{\infty}(\mathrm{End}(TE))$ which are zero near E^{crit} and can be written, in parallel with (2.1), as

$$Y_x = \begin{pmatrix} 0 & 0 \\ Y_x^{\mathrm{vh}} & Y_x^{\mathrm{vv}} \end{pmatrix} \tag{2.4}$$

with Y_x^{vv} an infinitesimal deformation of the compatible complex structure J_x^{vv}, and $Y_x^{\mathrm{vv}}J_x^{\mathrm{vh}} + J_x^{\mathrm{vv}}Y_x^{\mathrm{vh}} = -Y_x^{\mathrm{vh}}j_z$. There is a further requirement about Y near $\partial_{\mathrm{h}}E$, the linearization of (iv), which we leave to the reader to write down. The space $\mathscr{B}$ of sections $u: S \to E$ satisfying $u(\partial S) \subset Q$ is also an infinite-dimensional manifold, with $T\mathscr{B}_u = \{X \in C^{\infty}(u^*TE^{\mathrm{v}}): X_z \in T(Q_z) \text{ for all } z \in \partial S\}$; note that $u^*TE^{\mathrm{v}} \to S$ is really a vector bundle, since u as a smooth section of π avoids E^{crit}. Consider the infinite-dimensional vector bundle $\mathscr{E} \to \mathscr{B} \times \mathscr{J}(E,\pi,j)$ whose fibre at (u,J) is $\Omega^{0,1}(u^*TE^{\mathrm{v}})$, the space of $(0,1)$-forms on (S,j) with values in $u^*(TE^{\mathrm{v}}, J|TE^{\mathrm{v}})$. It has a canonical section $\bar{\partial}^{\mathrm{univ}}(u,J) = \frac{1}{2}(Du + J \circ Du \circ j)$, and the zero set $\mathscr{M}^{\mathrm{univ}} = (\bar{\partial}^{\mathrm{univ}})^{-1}(0)$ consists of pairs (u,J) such that u is (j,J)-holomorphic. We denote by $\bar{\partial}_J$, $\mathscr{M}_J$ the restrictions of $\bar{\partial}^{\mathrm{univ}}$, $\mathscr{M}^{\mathrm{univ}}$ to a fixed $J \in \mathscr{J}(E,\pi,j)$. The derivative of $\bar{\partial}_J$ at $u \in \mathscr{M}_J$ is a map $D_{u,J}: T\mathscr{B}_u \to \mathscr{E}_{u,J}$. An explicit formula is $D_{u,J}(X) = \frac{1}{2}(L_{\tilde{X}}J) \circ Du \circ j$ where $\tilde{X}$ is any section of TE^{v}, defined on a neighbourhood of $\mathrm{im}(u)$ in E, such that $u^*\tilde{X} = X$. After choosing a torsion-free connection ∇ on TE (away from E^{crit}) which preserves the integrable subbundle TE^{v}, one transforms this into the more familiar expression

$$D_{u,J}(X) = \bar{\partial}_{J,u^*\nabla}X + \tfrac{1}{2}(\nabla_X J) \circ Du \circ j, \tag{2.5}$$

in which $\bar{\partial}_{J,u^*\nabla} = (u^*\nabla)^{0,1}$ is the $\bar{\partial}$-operator associated to the pullback connection $u^*\nabla$ on $u^*(TE^{\mathrm{v}}, J|TE^{\mathrm{v}})$. The derivative of $\bar{\partial}^{\mathrm{univ}}$, $D_{u,J}^{\mathrm{univ}}: T\mathscr{B}_u \times T\mathscr{J}(E,\pi,j)_J \to \mathscr{E}_{u,J}$, is obviously

$$D_{u,J}^{\mathrm{univ}}(X,Y) = D_{u,J}(X) + \tfrac{1}{2}Y \circ Du \circ j. \tag{2.6}$$

From this point onwards, we pass to a more realistic situation, and assume that S is compact with $\partial S \neq \emptyset$. Our first observation is a consequence of (iv). Informally speaking, it says that for the purposes of pseudo-holomorphic sections, the boundary $\partial_{\mathrm{h}}E$ of the fibres can be ignored.

Lemma 2.2. *For every $J \in \mathscr{J}(E,\pi,j)$ there is a compact subset $K \subset E \setminus \partial_h E$ such that all $u \in \mathscr{M}_J$ satisfy $u(S) \subset K$.*

Proof. Let $W \subset E$ be a closed neighbourhood of $\partial_h E$ which, under the collar embedding provided by the Liouville vector fields on the fibres, corresponds to $[-\varepsilon;0]\times\partial_h E$ for some $\varepsilon > 0$. By definition $\sigma(W) = [-\varepsilon;0]$. After possibly making W and ε smaller, we may assume that $W \cap Q = \emptyset$, that $\Omega|TE^h$ vanishes on W, and that (iv) holds there. From $\Omega|TE^h_x = 0$ and $J(TE^h_x) = TE^h_x$ it follows that $\Omega(X, JX) \geqslant 0$ for all $X \in TE_x$, $x \in W$. Take $u \in \mathscr{M}_J$ and consider the function $h = e^{\sigma} \circ u$ on $U = u^{-1}(W)$. This satisfies $h|\partial U \equiv \mathrm{e}^{-\varepsilon}$ and is subharmonic, because $\mathrm{d}(\mathrm{d}h \circ j) = \mathrm{d}(\Theta \circ J \circ Du \circ j) = -u^*\Omega \leqslant 0$. It follows that $h \leqslant \mathrm{e}^{-\varepsilon}$ everywhere, which shows that $K = E \setminus \mathrm{int}(W)$ has the required property. □

The action of $u \in \mathscr{B}$ is defined to be $A(u) = \int_S u^*\Omega$. This is actually the same for all u, since

$$\int_S u^*\Omega = \int_{\partial S} u^*\Theta = \int_{\partial S} \kappa_Q. \tag{2.7}$$

Therefore, if one equips E with a symplectic form $\Omega + \pi^*\beta$ taming J, all the $u \in \mathscr{M}_J$ become pseudo-holomorphic curves with the same energy $\frac{1}{2}\int_S \|Du\|^2 = A(u) + \int_S \beta$.

Lemma 2.3. *$\mathscr{M}_J$ is compact in any C^r-topology.*

Proof. We apply the Gromov compactness theorem for (j,J)-holomorphic maps $S \to E$ with boundary in Q. The bubble components in the Gromov limit appear through a reparametrization which "magnifies" successively smaller parts of the domain. Since in our case all maps are sections, the bubbles are either nonconstant J-holomorphic spheres in some fibre E_z, or nonconstant J-holomorphic discs in E_z, $z \in \partial S$, with boundary on Q_z. But both are excluded by our assumptions, since Ω is exact and $Q_z \subset E_z$ an exact Lagrangian submanifold. □

A less formal version of the infinite-dimensional framework introduced above involves spaces of $W^{1,p}$-sections, $p > 2$. We omit the construction itself, and only mention its main consequence. For $(u,J) \in \mathscr{M}^{\mathrm{univ}}$, the differential operator $D_{u,J}$ extends to a Fredholm operator $\mathscr{W}^1_u \to \mathscr{W}^0_{u,J}$ from the $W^{1,p}$-completion $\mathscr{W}^1_u$ of $T\mathscr{B}_u$ to the L^p-completion $\mathscr{W}^0_{u,J}$ of $\mathscr{E}_{u,J}$. We denote this extension equally by $D_{u,J}$. If it is onto (in which case one says that u is regular), $\mathscr{M}_J$ is a smooth finite-dimensional manifold near u. If that holds for all $u \in \mathscr{M}_J$, J itself is called regular, and the space of such J is denoted by $\mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j) \subset \mathscr{J}(E,\pi,j)$.

Lemma 2.4. *$\mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j)$ is C^∞-dense in $\mathscr{J}(E,\pi,j)$. In fact the following stronger statement holds: take some nonempty open subset $U \subset S$ and a $J \in \mathscr{J}(E,\pi,j)$. Then there are $J' \in \mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j)$ arbitrarily close to J, such that $J = J'$ outside $\pi^{-1}(U)$.*

Proof. Even though this is a well-known argument, we recall part of it as a preparation for subsequent more refined versions. Fix U and J. By Lemmas 2.2 and 2.3 there is an open subset $V \subset E$ with $\bar{V} \cap (\partial_h E \cup E^{\mathrm{crit}}) = \emptyset$, such that $\mathrm{im}(u) \subset V$ for each $u \in \mathscr{M}_J$. A modified version of the compactness argument shows that this property, with the same V, remains true for all almost complex structures in $\mathscr{J}(E,\pi,j)$ which are sufficiently close to J. We want to make J regular by perturbing it on $V \cap \pi^{-1}(U)$. Take $\mathscr{T} \subset T\mathscr{J}(E,\pi,j)_J$ to be the subset of those Y which vanish outside $V \cap \pi^{-1}(U)$, and consider the operator

$$D^{\mathrm{univ}}_{u,J} : \mathscr{W}^1_u \times \mathscr{T} \to \mathscr{W}^0_{u,J} \tag{2.8}$$

given by the same formula as in (2.6). By a standard argument surjectivity of this operator implies the desired result (strictly speaking, what comes up in this argument is a dense subspace of those Y which have "finite C^∞_ε-norm", but since the first component $D_{u,J}$ is Fredholm, this makes no difference as far as surjectivity is concerned). Let $F \to S$ be the bundle dual to $\Lambda^{0,1}(u^*TE^{\mathrm{v}})$, so that $(\mathscr{W}^0_{u,J})^* \cong L^q(F)$, $p^{-1}+q^{-1}=1$. Suppose that $\eta \in L^q(F)$ is orthogonal to the image of (2.8). It then satisfies

$$D^*_{u,J}\eta = 0 \text{ on } S \setminus \partial S \quad \text{and} \quad \int_S \langle \eta, Y \circ Du \circ j\rangle = 0 \text{ for } Y \in \mathscr{T}. \tag{2.9}$$

The first equation implies that η is smooth away from the boundary. Suppose that $z \in U \setminus \partial S$ is a point where $\eta_z \neq 0$, and set $x = u(z) \in V \cap \pi^{-1}(U)$. Take a (j,J)-antilinear map $Z : TS_z \to TE^{\mathrm{v}}_x$ such that $\langle \eta_z, Z \circ j\rangle \neq 0$. One can see from (2.4) that there is a $Y \in T\mathscr{J}(E,\pi,j)_J$ with $Y^{\mathrm{vh}}_x = Z$, $Y^{\mathrm{vv}}_x = 0$, and this will satisfy $\langle \eta_z, (Y \circ Du \circ j)_z\rangle \neq 0$. By multiplying Y with a bump function supported near x, one can achieve that it lies in $\mathscr{T}$ and that $\int_S \langle \eta,\ Y \circ Du \circ j\rangle \neq 0$, a contradiction. This means that $\eta \,|\, (U \setminus \partial S) = 0$. By unique continuation $\eta \,|\, (S \setminus \partial S) = 0$, which proves that $\eta = 0$. □

Take some $\zeta \in \partial S$ and consider the map $\mathrm{ev}_\zeta : \mathscr{B} \to Q_\zeta$, $\mathrm{ev}_\zeta(u) = u(\zeta)$. The next result belongs to a type called "transversality of evaluation".

Lemma 2.5. *Let g be a smooth map from some arbitrary manifold G to Q_ζ. Then, for any J and U as in the previous lemma, there are $J' \in \mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j)$ arbitrarily close to J, with $J' = J$ outside $\pi^{-1}(U)$, such that $\mathrm{ev}_\zeta | \mathscr{M}_{J'}$ is transverse to g.*

Proof. In the same set-up as before, one now has to prove that for $u \in \mathscr{M}_J$ and $x = u(\zeta) \in Q_\zeta$, the operator

$$\mathscr{W}^1_u \times \mathscr{T} \to \mathscr{W}^0_{u,J} \times T(Q_\zeta)_x, \qquad (X,Y) \mapsto (D^{\mathrm{univ}}_{u,J}(X,Y), X_\zeta)$$

is onto. Take (η,ξ) orthogonal to the image, with $\xi \in T(Q_\zeta)^\vee_x$. One still has (2.9) and as before it follows that $\eta = 0$. Then $\langle \xi, X_\zeta\rangle = 0$ for all $X \in \mathscr{W}^1_u$, so that $\xi = 0$ as well. □

For a given (E,π,Q) and ζ, one now proceeds as follows. After choosing some j and a $J \in \mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j)$, one obtains a smooth compact moduli space $\mathscr{M}_J$; and then one sets

$$\Phi_1(E,\pi,Q) = (\mathrm{ev}_\zeta)_*[\mathscr{M}_J] \in H_*(Q_\zeta; \mathbb{Z}/2).$$

This is independent of the choice of j, J by a standard argument using parametrized moduli spaces. The same reasoning shows that it remains invariant under any "smooth deformation" of the geometric objects involved, that is to say of Q, Ω, Θ or of the fibration $\pi : E \to S$ itself, as long as one remains within the class of exact Lefschetz fibrations with Lagrangian boundary conditions. It seems pointless

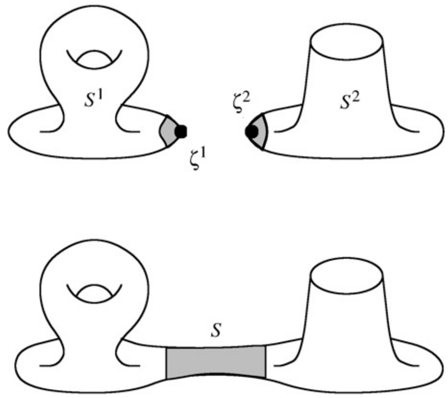

Fig. 2.

to formalize this notion of deformation, because the necessary conditions will be quite obviously satisfied in all our applications.

Remark 2.6. Since the regular spaces $\mathscr{M}_J$ are actual manifolds, one can refine $\Phi_1(E,\pi,Q)$ by regarding it as an element of the unoriented bordism group $MO_*(Q_\zeta)$. But even with this refinement, it is far from capturing all the information contained in $\mathscr{M}_J$. To get more sophisticated invariants one can use evaluation at several points, allowing those to move; we will now explain the simplest version of this. Given $\zeta \in \partial S$, choose a positively oriented path $c:[0;1] \to \partial S$, $c(0)=c(1)=\zeta$, parametrizing the boundary component on which ζ lies. Let $\phi_t : Q_{c(t)} \to Q_\zeta$ be the diffeomorphisms obtained from symplectic parallel transport along $c|[t;1]$. Denote by $\Delta, \Gamma \subset Q_\zeta^2$ the diagonal and the graph of ϕ_0, respectively. For regular J, the parametrized evaluation map

$$\tilde{\mathrm{ev}}_\zeta : [0;1] \times \mathscr{M}_J \to [0;1] \times Q_\zeta^2, \qquad \tilde{\mathrm{ev}}_\zeta(t,u) = (t, u(\zeta), \phi_t(u(c(t))))$$

represents a class $\tilde{\Phi}_2(E,\pi,Q) \in H_*([0;1] \times Q_\zeta^2,\ \{0\} \times \Gamma \cup \{1\} \times \Delta; \mathbb{Z}/2)$, and this (as well as its cobordism version) is an invariant of (E,π,Q). An example of this invariant will be computed in Remark 2.17(iii).

Let S^1, S^2 be two compact surfaces with marked points $\zeta^k \in \partial S^k$, and denote by $S = S^1 \#_{\zeta^1 \sim \zeta^2} S^2$ their boundary connected sum (Fig. 2). To be precise, one should choose oriented embeddings $\psi^k : \bar{D}^+(1) \to S^k$ of the closed half-disc $\bar{D}^+(1) = \{z \in \mathbb{C} : |z| \leqslant 1,\ \mathrm{im}\, z \geqslant 0\}$ into S^k, such that $\psi^k(0) = \zeta^k$ and $(\psi^k)^{-1}(\partial S^k) = [-1;1]$. Then, writing $D^+(\rho)$ for the open half-disc of some radius $0 < \rho < 1$, one forms S by taking the two $S^k \setminus \psi^k(D^+(\rho))$ and identifying $\psi^1(z)$ with $\psi^2(-\rho^{-1}z)$. Suppose that we have exact Lefschetz fibrations (E^k, π^k) over S^k, such that their symplectic connections are trivial on $\mathrm{im}(\psi^k)$; together with Lagrangian boundary conditions $Q^k \subset E^k$, such that the one-forms κ_{Q^k} vanish on $\psi^k([-1;1])$; and finally, an exact symplectic manifold M with an exact Lagrangian submanifold L, and isomorphisms $\phi^k : M \to (E^k)_{\zeta^k}$ satisfying $\phi^k(L) = (Q^k)_{\zeta^k}$. One can then glue the (E^k, π^k) and Q^k to an exact Lefschetz fibration over S. To do that, one first uses symplectic parallel transport to construct embeddings

$$\begin{tikzcd} \bar{D}^+(1) \cdot M \arrow[r, "\Psi^k"] \arrow[d] & E^k \arrow[d, "\pi^k"] \\ \bar{D}^+(1) \arrow[r, "\psi^k"] & S^k \end{tikzcd}$$

such that $\Psi^k | \{0\} \times M = \phi^k$. These will satisfy

$$(\Psi^k)^* \Omega^k = \omega, \qquad (\Psi^k)^* \Theta^k = \theta + \mathrm{d}R^k, \qquad (\Psi^k)^{-1}(Q^k) = [-1;1] \times L$$

for some functions R^k. As in the pasting construction described in Section 1.1, one needs to introduce modified forms $\tilde{\Theta}^k$ such that $(\Psi^k)^* \tilde{\Theta}^k = \theta$ near $\{0\} \times M$. The functions K_{Q^k} need to be modified accordingly, but that is rather straightforward, so we will not write it down explicitly. Now take the $E^k \setminus \Psi^k(D^+(\rho) \times M)$ and identify $\Psi^1(z,x)$ with $\Psi^2(-\rho z^{-1}, x)$. This yields a manifold E with a map $\pi : E \to S$, and the remaining data matches up, producing the structure of an exact Lefschetz fibration together with a Lagrangian boundary condition Q.

Assume that one has chosen complex structures j^k on S^k such that the ψ^k are holomorphic; these determine a complex structure j on S. Take $J^k \in \mathscr{J}(E^k, \pi^k, j^k)$ such that $(\Psi^k)^*(J^k)$ is the product of the standard complex structure on $\bar{D}^+(1)$ and of some fixed ω-compatible almost complex structure on M (the same for both k). Then there is a canonical induced $J \in \mathscr{J}(E, \pi, j)$. Note that even though we have restricted the behaviour of J^k over $\mathrm{im}(\Psi^k)$, it is still possible to choose them regular, by using the more precise statement in Lemma 2.4. Moreover, Lemma 2.5 says that for suitably chosen J^k, the evaluation maps $\mathrm{ev}_{\zeta^k} | \mathscr{M}_{J^k} : \mathscr{M}_{J^k} \to Q^k_{\zeta_k} \cong L$ will be transverse to each other.

Proposition 2.7. *Assume that $J^k \in \mathscr{J}^{\mathrm{reg}}(E^k, \pi^k, Q^k, j^k)$ for $k = 1, 2$, and that the $\mathrm{ev}_{\zeta_k} | \mathscr{M}_{J^k}$ are mutually transverse. Choose a sufficiently small parameter ρ for the gluing. Then $J \in \mathscr{J}^{\mathrm{reg}}(E, \pi, Q, j)$, and there is a diffeomorphism*

$$\mathscr{M}_J \cong \mathscr{M}_{J^1} \times_L \mathscr{M}_{J^2},$$

where the right-hand side is the fibre product of $(\mathrm{ev}_{\zeta^1}, \mathrm{ev}_{\zeta^2})$. □

This is an average specimen of the "gluing theorem" type. The closest related argument in the literature would seem to be the gluing theory for pseudo-holomorphic discs from [8, Section 18], which is far more sophisticated than what we need here; as an alternative, one can probably adapt the proof of the more familiar gluing theorem for closed pseudo-holomorphic curves [21, Section 6], [15, Appendix A], [14]. The obvious next step would be to write down the outcome as a "gluing formula". However, while gluing will be important later, the situation then will be slightly different from that covered by Proposition 2.7. For this reason, further discussion is postponed to Section 2.4.

2.2. Horizontality

Let (E, π) be an exact Lefschetz fibration. Choose a complex structure j on its base. $J \in \mathscr{J}(E, \pi, j)$ is called horizontal if $J_x(TE^{\mathrm{h}}_x) = TE^{\mathrm{h}}_x$ for all $x \notin E^{\mathrm{crit}}$, or what is equivalent, if $\Omega(\cdot, J\cdot)$ is symmetric. In terms of (2.1) these are just the J with $J^{\mathrm{vh}} = 0$, which shows that they form a contractible subspace $\mathscr{J}^{\mathrm{h}}(E,\pi,j) \subset \mathscr{J}(E,\pi,j)$. The importance of horizontal almost complex structures is that they are sensitive to the geometry of the symplectic connection; the following is a particularly simple instance of this.

Lemma 2.8. *Let (E,π) be an exact Lefschetz fibration over a compact surface S, $\partial S \neq \emptyset$, with a Lagrangian boundary condition Q. Assume that (E,π) has nonnegative curvature, and that the boundary condition satisfies $\int_{\partial S} \kappa_Q < 0$. Then $\Phi_1(E,\pi,Q) = 0$.*

Proof. For $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j)$, take a symplectic form $\Omega + \pi^*\beta$ as in Lemma 2.1. Then J is compatible with it in the ordinary sense of the word; we denote the associated metric by $\|\cdot\|$. Write $\Omega|TE^{\mathrm{h}} = f(\pi^*\beta|TE^{\mathrm{h}})$ with $f \in C^\infty(E \setminus E^{\mathrm{crit}},\mathbb{R})$. For any map u from a compact Riemann surface, possibly with boundary, to E there is the familiar equality $\frac{1}{2}\int \|Du\|^2 = \int u^*(\Omega + \pi^*\beta) + \int \|\bar{\partial}_J u\|^2$. Specialize to sections u and split $Du = (Du)^{\mathrm{h}} + (Du)^{\mathrm{v}}$ into horizontal and vertical parts. Since $\|(Du)^{\mathrm{h}}\|^2 = 2(f(u)+1)\beta$, one obtains

$$\frac{1}{2}\int_S \|(Du)^{\mathrm{v}}\|^2 + \int_S f(u)\beta = \int_S u^*\Omega + \int_S \|\bar{\partial}_J u\|^2. \tag{2.10}$$

This implies that $\mathscr{M}_J = \emptyset$. In fact, the curvature assumption is just that $f \geqslant 0$, while for $u \in \mathscr{M}_J$ one would have $\int_S u^*\Omega < 0$ by (2.7) and the second assumption. □

A section $u: S \to E$ is called horizontal if $Du_z(TS_z) = (TE^{\mathrm{h}})_{u(z)}$ for all $z \in S$. To see the geometric meaning of this, it is convenient to exclude temporarily the presence of critical points, so that (E,π) is an exact symplectic fibration. If u is horizontal, parallel transport along any path $c:[a;b] \to S$ carries $u(c(a)) \in E_{c(a)}$ to $u(c(b)) \in E_{c(b)}$. In other words, if M is some fibre of E and $x \in M$ the unique point through which u passes, the structure group of the symplectic connection on E is reduced from $\mathrm{Symp}^{\mathrm{e}}(M)$ to the subgroup $\mathrm{Symp}^{\mathrm{e}}(M,x)$ of maps preserving x. This entails a restriction on the curvature, namely, writing $\Omega|TE^{\mathrm{h}} = f(\pi^*\beta|TE^{\mathrm{h}})$ for some positive $\beta \in \Omega^2(S)$, one has

$$\mathrm{d}(f|E_z)_{u(z)} = 0 \quad \text{for all } z \in S. \tag{2.11}$$

If u is horizontal, the symplectic vector bundle $u^*TE^{\mathrm{v}} \to S$ has a preferred connection ∇^u, obtained by linearizing parallel transport around u. Equivalently, this is induced from the connection on E by the derivative map $\mathrm{Symp}^{\mathrm{e}}(M,x) \to \mathrm{Sp}(TM_x)$. Explicitly $\nabla^u_Z X = u^*([Z^{\mathrm{h}},\tilde{X}])$, where $\tilde{X}$ is any section of TE^{v} with $u^*\tilde{X} = X$. Using the canonical isomorphism $\mathrm{sp}(V) \cong \mathrm{sym}^2(V^*)$ for any symplectic vector space V, one can write the curvature of ∇^u as a two-form on S with values in quadratic forms on the fibres of u^*TE^{v}. In those terms it is given by $F_{\nabla^u} = \mathrm{Hess}(f|E_z)_{u(z)}\beta$, which is well defined by (2.11). We say that ∇^u is nonnegatively curved if all these Hessians are $\geqslant 0$. The "infinitesimal deformations" of a horizontal section u are the covariantly constant sections, $\nabla^u X = 0$. If such a section exists, it further reduces the structure group of (E,π) to the subgroup of maps in $\mathrm{Symp}^{\mathrm{e}}(M,x)$ which preserve a certain tangent vector at x. The resulting curvature restriction is

$$\mathrm{Hess}(f)_{u(z)}(X(z),X(z)) = 0 \quad \text{for all } z \in S. \tag{2.12}$$

Finally, if one readmits critical points, all the formulae derived above remain valid, with essentially the same proofs, except that when talking about the symplectic connection on E it is necessary to restrict to a neighbourhood of a fixed horizontal section.

The connection between the two notions which we have introduced so far is that *a horizontal section is (j,J)-holomorphic for any horizontal J*. This provides a useful class of pseudo-holomorphic sections with a geometric origin, but it also raises a problem: if the space of horizontal sections has "too large dimension", one cannot find an almost complex structure which is both regular and horizontal. The rest of this section discusses this issue in more detail. Assume from now on that S is compact with $\partial S \neq \emptyset$, and that we have a Lagrangian boundary condition Q. Write $\mathscr{M}^{\mathrm{h}}$ for the space of horizontal sections u which lie in $\mathscr{B}$, meaning that $u(\partial S) \subset Q$. Since a horizontal section is determined by its value at any point, evaluation at $\zeta \in \partial S$ identifies $\mathscr{M}^{\mathrm{h}}$ with a subset of Q_ζ. As observed above, $\mathscr{M}^{\mathrm{h}} \subset \mathscr{M}_J$ for all $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j)$.

Lemma 2.9. *Let $U \subset S$ be a nonempty open subset, such that any partial section $u: U \to E|U$ which is horizontal and satisfies $u(\partial S \cap U) \subset Q$ is the restriction of some $u' \in \mathscr{M}^{\mathrm{h}}$. Then, given some $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j)$, there are $J' \in \mathscr{J}^{\mathrm{h}}(E,\pi,j)$ arbitrarily close to it and which agree with it outside $\pi^{-1}(U)$, with the property that any $u \in \mathscr{M}_{J'} \setminus \mathscr{M}^{\mathrm{h}}$ is regular.*

Proof. Take $V \subset E$ as in the proof of Lemma 2.4. In parallel with the argument there, we have to show that for any $u \in \mathscr{M}_J \setminus \mathscr{M}^{\mathrm{h}}$,

$$D^{\mathrm{univ}}_{u,J} : \mathscr{W}^1_u \times \mathscr{T}^{\mathrm{h}} \to \mathscr{W}^0_{u,J} \tag{2.13}$$

is onto. Here $\mathscr{T}^{\mathrm{h}} = \mathscr{T} \cap T\mathscr{J}^{\mathrm{h}}(E,\pi,j)_J$ is the space of infinitesimal deformations Y of J within the class of horizontal almost complex structures, that is to say with $Y^{\mathrm{vh}} = 0$ in (2.4), and such that $Y = 0$ outside $V \cap \pi^{-1}(U)$. As before, an η which is orthogonal to the image of (2.13) is smooth away from the boundary and satisfies

$$\int_S \langle \eta, Y \circ Du \circ j \rangle = 0 \quad \text{for } Y \in \mathscr{T}^{\mathrm{h}}. \tag{2.14}$$

Suppose that $u|U$ is horizontal. By assumption one could then find a $u' \in \mathscr{M}^{\mathrm{h}}$ with $u'|U = u|U$; unique continuation for pseudo-holomorphic curves would imply that $u' = u$, a contradiction. Hence there is a $z \in U$ such that $Du_z(TS_z) \neq (TE^{\mathrm{h}})_{u(z)}$. This is an open condition, so we can assume that $z \in U \setminus \partial S$. By choosing Y suitably in $\mathscr{T}^{\mathrm{h}}$, one can make $(Y \circ Du \circ j)_z$ equal to any arbitrary (j,J)-antilinear homomorphism $TS_z \to (TE^{\mathrm{v}})_{u(z)}$. In particular, if $\eta_z \neq 0$ one can achieve that $\langle \eta_z, (Y \circ Du \circ j)_z \rangle \neq 0$, which after multiplying with a cutoff function leads to a contradiction with (2.14). The same argument applies to all points close to z, proving that η vanishes on a nonempty open subset, from which it follows that $\eta = 0$. □

In particular, taking $U = S$ shows that if $\mathscr{M}^{\mathrm{h}} = \emptyset$ then $\mathscr{J}^{\mathrm{reg,h}}(E,\pi,Q,j) = \mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j) \cap \mathscr{J}^{\mathrm{h}}(E,\pi,j) \subset \mathscr{J}^{\mathrm{h}}(E,\pi,j)$ is a dense subset. In the same way one proves the following analogue of Lemma 2.5.

Lemma 2.10. *Let g be a smooth map from an arbitrary manifold G to Q_ζ, for some $\zeta \in \partial S$. Let $U \subset S$ be a nonempty open subset, such that there are no horizontal partial sections $u: U \to E|U$ with $u(\partial S \cap U) \subset Q$. Given $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j)$, there are $J' \in \mathscr{J}^{\mathrm{reg,h}}(E,\pi,Q,j)$ arbitrarily close to it and which agree with it outside $\pi^{-1}(U)$, such that $\mathrm{ev}_\zeta|\mathscr{M}_{J'}$ is transverse to g.*

Let $T_{\mathrm{Zar}}(\mathscr{M}^{\mathrm{h}})_u$ be the Zariski tangent space of $\mathscr{M}^{\mathrm{h}}$ at some section u. It consists of those $X \in C^\infty(u^*TE^{\mathrm{v}})$ that satisfy $\nabla^u X = 0$ and lie in $T\mathscr{B}_u$, meaning that $X|\partial S \subset u^*(TQ \cap TE^{\mathrm{v}})$. The next result helps to determine when u, considered as a pseudo-holomorphic section for some horizontal almost complex structure, is regular.

Lemma 2.11. *Take $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j)$ and $u \in \mathscr{M}^{\mathrm{h}}$. Then $T_{\mathrm{Zar}}(\mathscr{M}^{\mathrm{h}})_u \subset \ker D_{u,J}$, and if moreover ∇^u is nonnegatively curved, the two spaces are equal.*

Proof. We take the second derivative of (2.10) at u. This is well defined because the action A is constant on $\mathscr{B}$ and the other terms vanish to first order at u. The outcome is that for $X \in T\mathscr{B}_u$,

$$\int_S \|\nabla^u X\|^2 + \int_S \mathrm{Hess}(f \circ u)(X,X)\beta = 2\int_S \|D_{u,J}X\|^2. \tag{2.15}$$

If $\nabla^u X = 0$ then $\mathrm{Hess}(f \circ u)(X,X) = 0$ by (2.12), which implies that $D_{u,J}X = 0$. If ∇^u has nonnegative curvature, the converse also holds, since the second term in (2.15) is $\geqslant 0$. □

Call $\mathscr{M}^{\mathrm{h}}$ clean if it is a smooth manifold and its tangent space is everywhere equal to $T_{\mathrm{Zar}}\mathscr{M}^{\mathrm{h}}$. The next result can be considered as a limiting case of Lemma 2.8 (and is again just a sample application, in itself without any great importance, but hopefully instructive).

Lemma 2.12. *Assume that (E,π) has nonnegative curvature, and that the Lagrangian boundary condition Q satisfies $\int_{\partial S} \kappa_Q = 0$. In addition, assume that $\mathscr{M}^{\mathrm{h}}$ is clean and that its dimension agrees at every point with the index of $D_{u,J}$. Then $\Phi_1(E,\pi,Q) = [\mathrm{ev}_\zeta(\mathscr{M}^{\mathrm{h}})]$.*

Proof. Take $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j)$, and consider (2.10). Since $f \geqslant 0$ and $\int_S u^*\Omega = 0$, it follows that $\mathscr{M}_J = \mathscr{M}^{\mathrm{h}}$, and moreover that every $u \in \mathscr{M}^{\mathrm{h}}$ satisfies $f(u) \equiv 0$, which in turn implies $\mathrm{Hess}(f|E_z)_{u(z)} \geqslant 0$. Applying Lemma 2.11 shows that $\dim\ker D_{u,J} = \dim T_{\mathrm{Zar}}\mathscr{M}^{\mathrm{h}} = \dim \mathscr{M}^{\mathrm{h}} = \mathrm{ind}\, D_{u,J}$, from which one sees that $\mathrm{coker}\, D_{u,J} = 0$; hence J is regular. □

2.3. A vanishing theorem

Let L be a framed exact Lagrangian sphere in an exact symplectic manifold M. According to Proposition 1.11 one can associate to it a standard fibration (E^L,π^L) over $\bar{D}(r)$ for some $r > 0$. Each fibre E^L_z, $z \neq 0$, contains a distinguished Lagrangian sphere Σ^L_z, described by Lemma 1.12; for $z = r$, the isomorphism $\phi^L : E^L_r \to M$ takes Σ^L_r to L. Define the standard Lagrangian boundary condition for (E^L,π^L) to be

$$Q^L = \bigcup_{z \in \partial\bar{D}(r)} \Sigma^L_z,$$
$$\kappa_{Q^L} = d^c(-\tfrac{1}{4}|z|)\,|\,\partial\bar{D}(r), \quad K_{Q^L} = 0. \tag{2.16}$$

This is indeed a Lagrangian boundary condition, as one can see from (1.20). Our aim here is to prove the following result about the invariant $\Phi_1(E^L,\pi^L,Q^L) \in H_*(\Sigma^L_r;\mathbb{Z}/2) \cong H_*(L;\mathbb{Z}/2)$:

Proposition 2.13. $\Phi_1(E^L,\pi^L,Q^L) = 0$ *for all M and L.*

The first part of the proof is a degeneration argument, in which one restricts the base to successively smaller discs. For $0 < s \leqslant r$ set $Q^{L,s} = \bigcup_{z \in \partial \bar{D}(s)} \Sigma_z^L$; together with $\kappa_{Q^{L,s}}$ and $K_{Q^{L,s}}$ as before, it is a Lagrangian boundary condition for $(E^{L,s}, \pi^{L,s}) = (E^L, \pi^L) \,|\, \bar{D}(s)$. This constitutes a deformation of (E^L, π^L, Q^L) in a suitable sense; which means that if one identifies Σ_r^L with Σ_s^L using parallel transport along $[s; r]$, then

$$\Phi_1(E^L, \pi^L, Q^L) = \Phi_1(E^{L,s}, \pi^{L,s}, Q^{L,s}) \tag{2.17}$$

for all s. Now fix some $J \in \mathscr{J}^{\mathrm{h}}(E^L, \pi^L, j)$, where j is the standard complex structure on $\bar{D}(r)$. Let $\mathscr{M}^{L,s}$ be the space of sections $\bar{D}(s) \to E^{L,s}$ which are holomorphic with respect to $j|\bar{D}(s)$ and $J|E^{L,s}$, and have boundary in $Q^{L,s}$. As $s \to 0$, $Q^{L,s}$ shrinks to the single critical point $\Sigma_0^L = \{x_0\}$ of π^L, and we would like to apply a compactness argument to elements of $\mathscr{M}^{L,s}$ in the limit. This looks a bit unpleasant as it stands, but one can modify the situation to make the degeneration of the $Q^{L,s}$ less singular, and then standard Gromov compactness is sufficient.

Lemma 2.14. *For some $0 < r' \leqslant r$, there is a compact almost complex manifold with corners $(\hat{E}^L, \hat{J})$, together with pseudo-holomorphic maps*

$$\begin{array}{ccc} \hat{E} & \xrightarrow{\eta^L} & E \\ {\scriptstyle \hat{\pi}^L}\big\downarrow & & \big\downarrow{\scriptstyle \pi^L} \\ \bar{D}(r') \cdot \bar{D}(1) & \xrightarrow{m} & \bar{D}(r') \end{array}$$

where $m(w,z) = wz$ is multiplication, such that the following properties are satisfied:

(i) *Away from $Z^L = (\eta^L)^{-1}(x_0) \cap (\hat{\pi}^L)^{-1}(0,0)$, the map η^L identifies $(\hat{E}^L, \hat{\pi}^L)$ with the pullback of (E^L, π^L) by m. In particular, restricting to any $w = s > 0$ gives a pseudo-holomorphic diffeomorphism*

$$\hat{E}^{L,s} = (\hat{\pi}^L)^{-1}(\{s\} \times \bar{D}(1)) \to E^{L,s}.$$

(ii) *$\hat{E}^L$ carries a symplectic form $\hat{\Omega}^L$ which tames its almost complex structure, and which is of the form*

$$\hat{\Omega}^L = (\eta^L)^* \Omega^L + (\hat{\pi}^L)^* \left(\frac{i}{2}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} + \frac{i}{2}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} + \mathrm{d}\gamma \right) + \delta,$$

with $\gamma = \frac{1}{4}\,\mathrm{im}((r' - w)\bar{z}\,\mathrm{d}z)$, and where δ is supported in a small neighbourhood of Z^L.

(iii) *$(\hat{E}^L, \hat{\Omega}^L)$ contains a Lagrangian submanifold with boundary $\hat{Q}^L$ which projects to $[0; r'] \times S^1 \subset \bar{D}(r') \times \bar{D}(1)$ and satisfies*

$$\eta^L(\hat{Q}^L \cap (\hat{\pi}^L)^{-1}(\{s\} \times S^1)) = \begin{cases} Q^{L,s}, & s > 0, \\ \{x_0\}, & s = 0. \end{cases}$$

It is convenient to begin with the local model for the construction. Choose some $r' > 0$ and set

$$E = \{x \in \mathbb{C}^{n+1} : \|x\| \leqslant 2\sqrt{r'},\ |q(x)| \leqslant r'\}, \qquad \Omega = \omega_{\mathbb{C}^{n+1}} | E,$$

$$\pi = q : E \to \bar{D}(r'), \qquad Q = \bigcup_{z \in \partial \bar{D}(r')} \Sigma_z, \tag{2.18}$$

where q is our standard quadratic function, and Σ_z is as in (1.11). In parallel with the reasoning above, one can also introduce $(E^s,\pi^s)=(E,\pi)|\bar{D}(s)$ and $Q^s=\bigcup_{z\in\partial\bar{D}(s)}\Sigma_z$ for $s\in(0;r']$. Take the pullback of (E,π) by the multiplication map, $m^*E=\{(w,z,x)\in\bar{D}(r')\times\bar{D}(1)\times E: q(x)=wz\}\subset\mathbb{C}^{n+3}$. This has one singular point $(0,0,0)$, which can be resolved by blowing it up inside $\mathbb{C}^{n+3}$ and taking the proper transform of m^*E. The outcome, which we denote by $\hat{E}$, comes with maps

$$\begin{tikzcd} \hat{E} \arrow[r, "\eta"] \arrow[d, "\hat{\pi}"'] & E \arrow[d, "\pi"] \\ \bar{D}(r')\cdot\bar{D}(1) \arrow[r, "m"] & \bar{D}(r'). \end{tikzcd}$$

(i$'$) η is a pullback map away from $Z=\eta^{-1}(0)\cap\hat{\pi}^{-1}(0,0)$.

That is obvious from the definition. Consider the two-form

$$\Omega+\frac{i}{2}\,\mathrm{d}w\wedge\mathrm{d}\bar{w}+\frac{i}{2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}+\mathrm{d}\gamma=\omega_{\mathbb{C}^{n+3}}+\mathrm{d}\gamma \tag{2.19}$$

on $\bar{D}(r')\times\bar{D}(1)\times E$, where γ is as in Lemma 2.14(ii). We claim that this is symplectic and tames the obvious complex structure. All one needs to verify is that $(\mathrm{d}\gamma)(\cdot,i\cdot)$ is nonnegative, which one can do by decomposing $\mathrm{d}\gamma=(\mathrm{d}\gamma)^{1,1}+(\mathrm{d}\gamma)^{0,2}+(\mathrm{d}\gamma)^{2,0}$; then $(\mathrm{d}\gamma)^{1,1}=\mathrm{re}(r'-w)\frac{i}{4}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}$ is nonnegative since $\mathrm{re}(r'-w)\geqslant 0$, while $(\mathrm{d}\gamma)^{0,2}(\cdot,i\cdot)$ vanishes because it is a two-form of type $(0,2)$, and similarly for $(\mathrm{d}\gamma)^{2,0}$. Now pull back (2.19) to the blowup, make it symplectic by adding a two-form δ supported near the exceptional divisor, and restrict that to $\hat{E}$. The outcome is:

(ii$'$) There is a symplectic form $\hat{\Omega}$ on $\hat{E}$ which tames the complex structure, and which is of the form $\eta^*\Omega+(\hat{\pi})^*(\frac{i}{2}\,\mathrm{d}w\wedge\mathrm{d}\bar{w}+\frac{i}{2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}+\mathrm{d}\gamma)+\delta$, with δ supported in a small neighbourhood of Z.

The subset

$$\{(w,z,x): w\in[0;r'],\ |z|=1,\ x\in\Sigma_{wz}\}\subset m^*E \tag{2.20}$$

is a submanifold, since it is the image of the embedding $S^1\times_{\mathbb{Z}/2}\bar{B}^{n+1}(1)\to\mathbb{C}^{n+3}$, $(z,y)\mapsto(|y|^2,z^2,zy)$. It is also Lagrangian with respect to (2.19); which is in fact the reason for our choice of symplectic form. Define $\hat{Q}$ to be the preimage of (2.20) in $\hat{E}$; provided that the support of δ has been chosen sufficiently small, this is again a Lagrangian submanifold. By construction one has

(iii$'$) $\hat{\pi}(\hat{Q})=[0;r]\times S^1$, and

$$\eta(\hat{Q}\cap\hat{\pi}^{-1}(\{s\}\times S^1))=\begin{cases} Q^s, & s>0,\\ \{0\}, & s=0.\end{cases}$$

Proof of Lemma 2.14. Take a holomorphic Morse chart (ξ,Ξ) around $x_0\in E_0^L$ as provided by Lemma 1.12(ii). After restricting the domains, we may assume that ξ is the inclusion $\bar{D}(r')\hookrightarrow\bar{D}(r)$ for some $0<r'\leqslant r$, that Ξ is defined on the set $E\subset\mathbb{C}^{n+1}$ from (2.18), and that $Q^{L,s}=\Xi(Q^s)$ for all

$0 < s \leqslant r'$. Construct a diagram

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] \arrow[rr, bend left, "\eta^L"] & m^*E \arrow[r] \arrow[d, "m^*\Xi"] & E \arrow[d, "\Xi"] \\
\hat{E}^L \arrow[r] \arrow[dr, "\hat{\pi}^L"'] & m^*E^L \arrow[r] \arrow[d] & E^{L,r'} \arrow[d, "\hat{\pi}^{L,r'}"] \\
& \bar{D}(r') \cdot \bar{D}(1) \arrow[r, "m"] & \bar{D}(r')
\end{tikzcd}$$

as follows: pull back E^L by multiplication m; the map $m^*\Xi$ induced by Ξ identifies neighbourhoods of the singular points $(0,0,0) \in m^*E$ and $(0,0,x_0) \in m^*E^L$; which means that there is a resolution

$$\hat{E}^L \to m^*E^L$$

modelled locally on $\hat{E} \to m^*E$. The given almost complex structure J induces one on m^*E, for which $m^*\Xi$ is holomorphic; this and the complex structure on $\hat{E}$ define the almost complex structure $\hat{J}$. Part (i) of the lemma now follows from the corresponding statement (i$'$) in the local model. Next, take the two-form $\Omega^L + \frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w} + \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z} + \mathrm{d}\gamma$ on m^*E^L; due to the nonnegative curvature of standard fibrations, see Lemma 1.12(ii), and to the fact that J is horizontal, this will tame the almost complex structure away from the singular point. One gets $\hat{\Omega}^L$ by gluing this together with the symplectic form $\hat{\Omega}$ from the local model, and that proves (ii). Similarly, the definition of $\hat{Q}^L$ follows that of $\hat{Q}$, and (iii$'$) implies (iii). □

Returning to the moduli spaces $\mathscr{M}^{L,s}$ of pseudo-holomorphic sections, we can now carry out the compactness argument mentioned above:

Lemma 2.15. *Choose some neighbourhood of the critical point $x_0 \in E^L$. For sufficiently small s, the image of all $u \in \mathscr{M}^{L,s}$ will lie in that neighbourhood.*

Proof. Let (s_k) be a sequence in $(0; r']$ converging to zero, and $u_k \in \mathscr{M}^{L,s_k}$ a corresponding sequence of sections. By Lemma 2.14(i), there is for each k a unique pseudo-holomorphic map

$$\hat{u}_k : \bar{D}(1) \to \hat{E}^L$$

with $\eta^L(\hat{u}_k(z)) = u_k(s_k z)$ and $\hat{\pi}^L(\hat{u}_k(z)) = (s_k, z)$. By part (iii) of the same lemma, this maps S^1 to $\hat{Q}^L$. Moreover, its energy is independent of k, since

$$\begin{aligned}\int_{\bar{D}(1)} \hat{u}_k^* \hat{\Omega}^L &= \int_{\bar{D}(s_k)} u_k^* \Omega^L + \int_{\bar{D}(1)} \frac{i}{2} \mathrm{d}z \wedge \mathrm{d}\bar{z} + \int_{\bar{D}(1)} \frac{i}{4}(r' - s_k)\mathrm{im}(\mathrm{d}\bar{z} \wedge \mathrm{d}z) + \int_{\bar{D}(1)} \hat{u}_k^* \delta \\ &= \frac{\pi}{2} s_k + \pi + \frac{\pi}{2}(r' - s_k) + 0 = \pi\left(1 + \frac{r'}{2}\right).\end{aligned}$$

Here we have used (1.20) for the first term; and the last term is zero because, without changing its cohomology class, one can modify δ to make its support arbitrarily close to Z, in which case, since u_k avoids the critical point x_0, $\hat{u}_k$ would not meet the support of δ. One can apply Gromov compactness to the sequence $\hat{u}_k : (\bar{D}(1), S^1) \to (\hat{E}^L, \hat{Q}^L)$; the limit of some subsequence will be a pseudo-holomorphic "cusp disc" or "stable disc" $\hat{u}_\infty$. Since $\mathrm{im}(\hat{u}_k) \subset (\hat{\pi}^L)^{-1}(\{s_k\} \times \bar{D}(1))$, we have $\mathrm{im}(\hat{u}_\infty) \subset (\hat{\pi}^L)^{-1}(\{0\} \times \bar{D}(1))$. Composing with η^L yields a "cusp disc" u_∞ in E whose boundary lies in $\eta^L(\hat{Q}^L \cap (\hat{\pi}^L)^{-1}(\{0\} \times S^1)) = \{x_0\}$. Because Ω^L is exact, u_∞ is necessarily constant equal to x_0. On the other hand, the image of u_k, for k large and in our subsequence, lies in an arbitrarily small neighbourhood of the image of u_∞. The rest is straightforward. □

The second part of the proof of Proposition 2.13 is an explicit computation in the local model (E, π) from (2.18). Note that while this is not an exact Lefschetz fibration, it still makes sense to consider the spaces $\mathscr{M}^s$ of holomorphic sections $w : \bar{D}(s) \to E^s$ with boundary in Q^s. In fact Q^s is a Lagrangian submanifold of $\mathbb{C}^{n+1}$, as one can see from (1.12), and the maximum principle ensures that any holomorphic disc in $\mathbb{C}^{n+1}$ with boundary in Q^s is actually contained in E^s.

Lemma 2.16. *$\mathscr{M}^s$ consists of maps $w(z) = s^{-1/2}az + s^{1/2}\bar{a}$, where $a \in \mathbb{C}^{n+1}$ satisfies $q(a) = 0$ and $\|a\|^2 = \frac{1}{2}$. All these maps are regular, in the sense that the associated Fredholm operators are surjective.*

Proof. The holomorphic functions $v : \bar{D}(s) \to \mathbb{C}$ which satisfy $v(z) \in z^{1/2}\mathbb{R}$ for all $z \in \partial\bar{D}(s)$ are $v(z) = s^{-1/2}cz + s^{1/2}\bar{c}$, for $c \in \mathbb{C}$. All components of $w \in \mathscr{M}^s$ must be of this form, and the conditions on a come from $q(w(z)) = z$. As for regularity, since we are dealing with the standard complex structure J_0 on $\mathbb{C}^{n+1}$, D_{w,J_0} is an actual $\bar{\partial}$-operator, see (2.5). Its kernel consists of holomorphic maps $X : \bar{D}(s) \to \mathbb{C}^{n+1}$ such that $X(z) \in \sqrt{z}\mathbb{R}^n$ for $z \in \partial\bar{D}(s)$, and $Dq(w(z))X(z) = 2\sum_k w_k(z)X_k(z) = 0$ for all z. The same argument as before determines all such X explicitly, the outcome being that $\ker D_{w,J_0} \cong \mathbb{R}^{2n-1}$. Using the Riemann–Roch formula for surfaces with boundary one computes that $\mathrm{ind}\, D_{w,J_0} = 2n - 1$, which shows that the cokernel is zero. □

The condition on a can be written as

$$\|\mathrm{re}\, a\|^2 = \|\mathrm{im}\, a\|^2 = \tfrac{1}{4}, \qquad \langle \mathrm{re}\, a, \mathrm{im}\, a\rangle = 0$$

so that $\mathscr{M}^s$ can be identified with the sphere bundle $S(T^*S^n)$ by mapping a to $(u, v) = (-2\,\mathrm{im}\, a,\ 2\,\mathrm{re}\, a)$. With this and the diffeomorphism $S^n \to \Sigma_s$, $x \mapsto \sqrt{s}x$, one can identify the evaluation $ev_s : \mathscr{M}^s \to \Sigma_s$ with the projection to the base $S(T^*S^n) \to S^n$. This clearly represents the zero cycle in $H_*(S^n; \mathbb{Z}/2)$. Hence, if one defined an invariant $\Phi_1(E^s, \pi^s, Q^s)$ in this local model, it would vanish.

Proof of Proposition 2.13. As in the proof of Lemma 2.15, we consider a holomorphic Morse chart (ξ, Ξ) with Ξ defined on $E \subset \mathbb{C}^{n+1}$ for some $r' > 0$. We know that for some sufficiently small $s > 0$, all $u \in \mathscr{M}^{L,s}$ lie in $\mathrm{im}(\Xi)$. Recall that Ξ is holomorphic with respect to the standard complex structure J_0 on E and to J on E^L; and that it takes Q^s to $Q^{L,s}$. This implies that composition with Ξ yields a diffeomorphism

$$\begin{array}{ccc} \mathscr{M}^S & \xrightarrow{\cong} & \mathscr{M}^{L,s} \\ \downarrow{\scriptstyle ev_s} & & \downarrow{\scriptstyle ev_s} \\ \Sigma_S & \xrightarrow{\Xi} & \Sigma_S^L. \end{array}$$

It is easy to see that, for $w \in \mathcal{M}^s$ and $u = \Xi \circ w \in \mathcal{M}^{L,s}$, the kernels and cokernels of the associated operators D_{w,J_0}, $D_{u,J}$ coincide. This shows that $J|E^{L,s}$ is regular, and that the evaluation cycle $\mathcal{M}^{L,s} \to \Sigma_s^L$ can be identified with projection $S(T^*S^n) \to S^n$, which as observed before is zero in homology; together with (2.17) this completes the argument. $\square$

Remark 2.17. (i) As pointed out in Remark 2.6, Φ_1 can be refined to an invariant taking values in unoriented bordism. Proposition 2.13 also applies to this refinement, since (2.17) comes from a cobordism of moduli spaces and the projection $S(T^*S^n) \to S^n$ is obviously cobordant to zero.

(ii) Suppose that (E^1, π^1, Q^1) is a standard fibration with its standard boundary condition, and that we have another exact Lefschetz fibration (E^2, π^2, Q^2), such that the two can be glued together to (E, π, Q) as in Section 2.1. Proposition 2.13 together with Proposition 2.7 implies that $\Phi_1(E, \pi, Q)$ will then be zero. By pushing this reasoning a little further, one arrives at the following result: let (E, π) be an arbitrary exact Lefschetz fibration over a compact base S, with Lagrangian boundary condition Q. Assume that there is a path $c : [a; b] \to S$ with $c(a) \in \partial S$, $c^{-1}(S^{\mathrm{crit}}) = \{b\}$, and $c'(b) \neq 0$, whose vanishing cycle $V_{c(a)}$ is isotopic to $Q_{c(a)}$ as an exact Lagrangian submanifold in $E_{c(a)}$; then $\Phi_1(E, \pi, Q) = 0$.

(iii) Despite the vanishing of their Φ_1-invariant, the moduli spaces of pseudo-holomorphic sections of a standard fibration are never empty. To see this, one has to use the invariant $\tilde{\Phi}_2$ mentioned in Remark 2.6. The computation of this can be reduced to the local model in the same way as before, using Lemma 2.15. The relevant parametrized evaluation map then becomes

$$
\begin{aligned}
&[0; 1] \times \mathcal{M}^s \to [0; 1] \times \Sigma_s \times \Sigma_s, \\
&(t, w) \mapsto (t, w(s), e^{\pi i(1-t)} w(s e^{2\pi i t})).
\end{aligned}
\tag{2.21}
$$

Here $\Sigma_{se^{2\pi it}} \to \Sigma_s$, $x \mapsto e^{\pi i(1-t)} x$ arises as parallel transport along $\partial \bar{D}(s)$ in positive direction. Using Lemma 2.16 one can make (2.21) even more concrete, identifying it with

$$
\begin{aligned}
&[0; 1] \times S(T^*S^n) \to [0; 1] \times S^n \times S^n, \\
&(t, u, v) \mapsto (t, v, -\cos(\pi t) v - \sin(\pi t) u).
\end{aligned}
$$

It is now easy to see that $\tilde{\Phi}_2(E^L, \pi^L, Q^L)$ is nonzero: it is the image of the fundamental class $[S^n \times S^n]$ under the map

$$
\begin{aligned}
&H_*(S^n \times S^n; \mathbb{Z}/2) \to H_*(S^n \times S^n, \Delta \cup \bar{\Delta}; \mathbb{Z}/2) \\
&\quad \cong H_*([0; 1] \times S^n \times S^n, \{0\} \times \bar{\Delta} \cup \{1\} \times \Delta; \mathbb{Z}/2)
\end{aligned}
$$

where Δ, $\bar{\Delta}$ are the diagonal and antidiagonal, respectively.

2.4. *Relative invariants*

A surface with strip-like ends is an oriented connected surface S, together with finite sets I^-, I^+ and oriented proper embeddings $\{\gamma_e : \mathbb{R}^- \times [0; 1] \to S\}_{e \in I^-}$, $\{\gamma_e : \mathbb{R}^+ \times [0; 1] \to S\}_{e \in I^+}$, such that $\gamma_e^{-1}(\partial S) = \mathbb{R}^{\pm} \times \{0; 1\}$. The images of the γ_e (the ends of S) should be mutually disjoint, and the complement of the union of all ends should be a relatively compact subset of S. We will always assume that there is at least one end.

Define an *exact Lefschetz fibration trivial over the ends of S* to be an exact Lefschetz fibration (E,π), whose regular fibres are isomorphic to some exact symplectic manifold M, together with smooth trivializations $\{\Gamma_e : \mathbb{R}^- \times [0;1] \times M \to \gamma_e^* E\}_{e\in I^-}$, $\{\Gamma_e : \mathbb{R}^+ \times [0;1] \times M \to \gamma_e^* E\}_{e\in I^+}$, such that $\Gamma_e^*\Omega$, $\Gamma_e^*\Theta$ are equal to the pullbacks of ω,θ by the projection $\mathbb{R}^\pm \times [0;1] \times M \to M$. When considering a Lagrangian boundary condition Q for such an exact Lefschetz fibration, we will always impose the additional condition that κ_Q vanishes on $\mathrm{im}(\gamma_e)\cap\partial S$. To see the significance of this, recall that the family $Q_z \subset E_z$ of Lagrangian submanifolds is preserved by symplectic parallel transport along ∂S. Since the symplectic connection is trivial on the ends, it follows that $\Gamma_e^{-1}(Q) = (\mathbb{R}^\pm \times \{0\} \times L_{e,0}) \cup (\mathbb{R}^\pm \times \{1\} \times L_{e,1})$ for some pair of Lagrangian submanifolds $L_{e,0}, L_{e,1} \subset M$; we say that Q is modelled on $(L_{e,0}, L_{e,1})$ over the end e. Using the definition of a Lagrangian boundary condition, one finds that

$$\mathrm{d}(K_Q \circ \Gamma_e) \,|\, \mathbb{R}^\pm \times \{k\} \times L_{e,k} = (\theta | L_{e,k}) - (\gamma_e^* \kappa_Q | \mathbb{R}^\pm \times \{k\}).$$

Our assumption is that the second term on the right-hand side vanishes, which implies that $K_Q(\Gamma_e(s,k,y))$ is independent of s. It follows that $L_{e,k}$ is an exact Lagrangian submanifold in a canonical way, with associated function $K_{L_{e,k}}(y) = K_Q(\Gamma_e(s,k,y))$.

In this situation, and under the additional assumption that the intersections $L_{e,0} \cap L_{e,1}$ are transverse for all e, we will associate to (E,π) and Q a relative invariant, which is a map between Floer cohomology groups

$$\Phi_0^{\mathrm{rel}}(E,\pi,Q) : \bigotimes_{e\in I^+} \mathrm{HF}(L_{e,0}, L_{e,1}) \to \bigotimes_{e\in I^-} \mathrm{HF}(L_{e,0}, L_{e,1}).$$

This is a modified version of the frameworks described in [24,19,25]. The fact that Φ_0^{rel} goes from positive to negative ends has to do with the use of Floer *co*homology, which we think of as behaving contravariantly. This is of course largely a matter of convention. In any case, there is no real difference between positive and negative ends, as one can be turned into the other by switching from $\gamma_e(s,t)$ to $\gamma_e(-s,1-t)$. Doing that does not change the relative invariant, up to the "Poincaré duality" isomorphism $\mathrm{HF}(L_{e,0}, L_{e,1}) \cong \mathrm{HF}(L_{e,1}, L_{e,0})^\vee$. One could therefore formulate the theory using only one kind of ends, bringing it closer to a (1+1)-dimensional TQFT. Finally, we should mention that the transverse intersection condition can be lifted. This goes by a standard argument, using the same "continuation map" technique as the proof of isotopy invariance of Floer cohomology. We will not discuss that further, since it is not necessary for our immediate purpose.

To begin with, a brief review of Floer cohomology, just to remind the reader of the special features of the "exact" situation. Let M be an exact symplectic manifold and (L_0, L_1) a pair of transversally intersecting exact Lagrangian submanifolds. The action functional on the path space $\mathscr{P}(L_0,L_1) = \{c \in C^\infty([0;1], M) : c(0) \in L_0,\ c(1) \in L_1\}$ is

$$a_{L_0,L_1}(c) = -\int c^*\theta + K_{L_1}(c(1)) - K_{L_0}(c(0))$$

(as pointed out to me by Oh, this form of the action functional appears in [17], and is then used crucially in his papers with Milinkovic). Its critical points are the constant paths c_x at points $x \in L_0 \cap L_1$. We write $a_{L_0,L_1}(x) = a_{L_0,L_1}(c_x) = K_{L_1}(x) - K_{L_0}(x)$ for their action. The Floer cochain space $\mathrm{CF}(L_0,L_1)$ is the vector space over $\mathbb{Z}/2$ with a basis given by these points; we denote the canonical basis vectors by $\langle x \rangle$. Let $\mathscr{J}(M)$ be the space of smooth families $J = (J_t)_{0 \leqslant t \leqslant 1}$ of almost complex structures on

M, each of which is ω-compatible and convex near the boundary. For $J \in \mathscr{J}(M)$ and $x_\pm \in L_0 \cap L_1$, one considers the space $\mathscr{F}_J(x_-, x_+)$ of maps

$$\sigma = \sigma(s,t)\colon \mathbb{R} \times [0;1] \to M,$$

$$\sigma(\mathbb{R} \times \{0\}) \subset L_0,\ \ \sigma(\mathbb{R} \times \{1\}) \subset L_1,$$

$$\partial\sigma/\partial t = J_t \partial\sigma/\partial s,$$

$$\lim_{s \to \pm\infty} \sigma(s, \cdot) = c_{x_\pm}, \tag{2.22}$$

where the limit is understood to be in the C^1-topology on $\mathscr{P}(L_0, L_1)$. There is a natural $\mathbb{R}$-action on $\mathscr{F}_J(x_-, x_+)$ by translation in s-direction. We denote by $\mathscr{F}_J^*(x_-, x_+)$ the subspace of maps which are not $\mathbb{R}$-invariant. Solutions σ of (2.22) can be interpreted as negative gradient flow lines for a_{L_0,L_1} in an L^2-metric on $\mathscr{P}(L_0, L_1)$; of course, this implies that $\mathscr{F}_J^*(x_-, x_+) = \emptyset$ whenever $a_{L_0,L_1}(x_-) \leqslant a_{L_0,L_1}(x_+)$. To each $\sigma \in \mathscr{F}_J(x_-, x_+)$ one can associate an operator $D_{\sigma,J}$ which linearizes (2.22), and which is Fredholm in suitable Sobolev spaces. There is a dense subspace $\mathscr{J}^{\mathrm{reg}}(M, L_0, L_1) \subset \mathscr{J}(M)$ of almost complex structures J for which all $D_{\sigma,J}$ are onto, and then the spaces $\mathscr{F}_J(x_-, x_+)$, as well as the quotients $\mathscr{F}_J^*(x_-, x_+)/\mathbb{R}$, are smooth manifolds. In that case one defines $n_J(x_-, x_+) \in \mathbb{Z}/2$ to be the number mod 2 of isolated points in $\mathscr{F}_J^*(x_-, x_+)/\mathbb{R}$. The map

$$d_J\langle x_+\rangle = \sum_{x_-} n_J(x_-, x_+)\langle x_-\rangle$$

has square zero, making $\mathrm{CF}(L_0, L_1)$ into a differential vector space. $\mathrm{HF}(L_0, L_1)$ is defined to be its cohomology $\ker d_J / \mathrm{im}\, d_J$. Unlike d_J itself, Floer cohomology can be shown to be independent of J up to canonical isomorphism. It is also invariant under deformations of L_0 or L_1 as exact Lagrangian submanifolds.

We now start constructing the relative invariant associated to (E, π) and Q. Choose a point $x_e \in L_{e,0} \cap L_{e,1}$ for each end e, and write $\mathscr{B}(\{x_e\})$ for the space of smooth sections $u : S \to E$ such that $u(\partial S) \subset Q$, and which over the ends have the form $\Gamma_e^{-1} \circ u \circ \gamma_e(s,t) = (s, t, \sigma_e(s,t))$ with maps $\sigma_e : \mathbb{R}^\pm \times [0;1] \to M$ satisfying $\lim_{s\to\pm\infty} \sigma_e(s, \cdot) = c_{x_e}$, in the same sense as in (2.22). The action integral $A(u) = \int_S u^*\omega$ is convergent for all $u \in \mathscr{B}(\{x_e\})$, and one gets a formula analogous to (2.7):

$$A(u) = \sum_{e \in I^-} a_{L_{e,0},L_{e,1}}(x_e) - \sum_{e \in I^+} a_{L_{e,0},L_{e,1}}(x_e) + \int_{\partial S} \kappa_Q. \tag{2.23}$$

Take a complex structure j on S (equal to j_0 near S^{crit} as always) which, on the ends, is induced from the standard complex structure $j_\pm$ on $\mathbb{R}^\pm \times [0;1]$. Choose also a $J_e = (J_{e,t})_{0 \leqslant t \leqslant 1} \in \mathscr{J}(M)$ for each e. Then $\mathscr{J}(E, \pi, j, \{J_e\})$ denotes the contractible space of almost complex structures J on E which are compatible relative to j, and such that $\Gamma_e^* J$, for each e, is the almost complex structure on $\mathbb{R}^\pm \times [0;1] \times M$ given by $j_\pm \times J_{e,t}$ at a point (s, t, y). For any such J, we denote by $\mathscr{M}_J(\{x_e\}) \subset \mathscr{B}(\{x_e\})$ the subspace of sections u which are (j, J)-holomorphic. The formal picture from Section 2.1 still applies: one has the vector bundle $\mathscr{E}_J \to \mathscr{B}(\{x_e\})$, its canonical section $\bar\partial_J$

whose zero-set is $\mathscr{M}_J(\{x_e\})$, as well as their "universal" versions $\mathscr{E}^{\mathrm{univ}}$, $\bar{\partial}^{\mathrm{univ}}$. Turning this picture into an analytically realistic one is more complicated than in the compact case. Still, the procedure is by now standard, and so we will not say more about it, except to mention that the derivative $D_{u,J}$ of $\bar{\partial}_J$ again extends to a Fredholm operator from a suitably defined $W^{1,p}$-version $\mathscr{W}^1_u$ of $T\mathscr{B}(\{x_e\})_u$ to an L^p-version $\mathscr{W}^0_{u,J}$ of $\mathscr{E}_{u,J}$. This allows one to define the notion of regularity of $u \in \mathscr{M}_J(\{x_e\})$, and of $J \in \mathscr{J}(E,\pi,j,\{J_e\})$, in the same way as before. The subspace of regular J is denoted by $\mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j,\{J_e\})$.

Remark 2.18. It is maybe helpful to mention that Floer's equation (2.22) itself can be made to fit into this framework. Namely, take the surface $S = \mathbb{R}\times[0;1]$ and the trivial exact symplectic fibration $\pi: E = S\times M \to S$, with the Lagrangian boundary condition $Q = (\mathbb{R}\times\{0\}\times L_0)\cup(\mathbb{R}\times\{1\}\times L_1)$; it is a tautology that this is modelled over the two ends of S on (L_0, L_1). Take the standard complex structure j on S, the same $J_e \in \mathscr{J}(M)$ for both ends e, and define $J \in \mathscr{J}(E,\pi,j,\{J_e\})$ to be the product $j \times J_{e,t}$ at a point (s,t,y). Sections $u: S \to E$ with $u(\partial S) \subset Q$ are of the form $u(s,t) = (s,t,\sigma(s,t))$, where $\sigma: \mathbb{R}\times[0;1] \to M$ is a map satisfying the boundary condition in (2.22). Moreover, u is (j,J)-holomorphic iff $\partial\sigma/\partial t = J_{e,t}\partial\sigma/\partial s$, so that one can identify $\mathscr{M}_J(x_-,x_+) = \mathscr{F}_{J_e}(x_-,x_+)$ for all $x_\pm$.

In parallel with the exposition in Section 2.1, we will now discuss the basic properties of the spaces $\mathscr{M}_J(\{x_e\})$. The next two results are analogues of Lemmas 2.2 and 2.4, and their proofs are the same.

Lemma 2.19. *For every $J \in \mathscr{J}(E,\pi,j,\{J_e\})$ there is a closed subset $K \subset E\backslash\partial_{\mathrm{h}}E$ such that $\pi|K: K \to S$ is proper, and which over the ends has the form $\Gamma_e^{-1}(K) = \mathbb{R}^\pm \times [0;1]\times K_e$ for some compact $K_e \subset M\setminus\partial M$, such that $u(S)\subset K$ for all $u \in \mathscr{M}_J(\{x_e\})$ and all points $\{x_e\}$.*

Lemma 2.20. *$\mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j,\{J_e\})$ is C^∞-dense in $\mathscr{J}(E,\pi,j,\{J_e\})$. More precisely, given some nonempty open subset $U \subset S$ which is disjoint from the ends, and a $J \in \mathscr{J}(E,\pi,j,\{J_e\})$, there are $J' \in \mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j,\{J_e\})$ arbitrarily close to J, such that $J = J'$ outside $\pi^{-1}(U)$.*

The remaining issue is compactness. After taking a symplectic form $\Omega + \pi^*\beta$ on E as in Lemma 2.1, and the associated metric, one finds that $\frac{1}{2}\int_W \|Du\|^2 \leqslant A(u) + \int_W \beta$ for any compact subset $W \subset S$ and $u \in \mathscr{M}_J(\{x_e\})$. Repeating the argument in Lemma 2.3, one derives from this that any sequence (u_i) in $\mathscr{M}_J(\{x_e\})$ has a subsequence which is C^r-convergent on compact subsets. It is necessary to go beyond this somewhat coarse result, and to study sequences (u_i) in the more appropriate *Gromov–Floer topology*. A compactification $\bar{\mathscr{M}}_J(\{x_e\})$ of $\mathscr{M}_J(\{x_e\})$ in this topology can be constructed by adding "broken sections". Since a very similar notion is part of the standard analytical package underlying Floer theory, we will not define the topology, and only describe the compactification as a set. Each point of it consists of

(i) the "principal component" $u \in \mathscr{M}_J(\{\hat{x}_e\})$ for some $\{\hat{x}_e\}$;
(ii) for each end $e \in I^\pm$, a finite sequence of points $\hat{x}_{e,0},\dots,\hat{x}_{e,l_e} \in L_{e,0}\cap L_{e,1}$, $l_e \geqslant 0$. If the end is negative (positive), this should satisfy $\hat{x}_{e,0} = x_e$ and $\hat{x}_{e,l_e} = \hat{x}_e$ ($\hat{x}_{e,0} = \hat{x}_e$ and $\hat{x}_{e,l_e} = x_e$);
(iii) Floer flow lines $\sigma_{e,m} \in \mathscr{F}^*_{J_e}(\hat{x}_{e,m-1},\hat{x}_{e,m})/\mathbb{R}$, for each e and $1 \leqslant m \leqslant l_e$.

Suppose that a sequence (u_i) converges to such a limit. One sees from (2.23) that for all i,

$$A(u_i) = A(u) + \sum_e \sum_{1 \leqslant m \leqslant l_e} (a_{L_{e,0},L_{e,1}}(\hat{x}_{e,m-1}) - a_{L_{e,0},L_{e,1}}(\hat{x}_{e,m})). \tag{2.24}$$

Note that all terms of the $\sum$ are > 0; therefore $A(u) \leqslant A(u_i)$, with equality iff $l_e = 0$ for all ends e. We will need another piece of information about the limit, which can be derived from the definition of the Gromov–Floer topology and a "gluing theorem" for linear elliptic operators. Namely, for $i \gg 0$,

$$\operatorname{ind} D_{u_i,J} = \operatorname{ind} D_{u,J} + \sum_e \sum_{1 \leqslant m \leqslant l_e} \operatorname{ind} D_{\sigma_{e,m},J_e}. \tag{2.25}$$

From now on assume that $J_e \in \mathscr{J}^{\mathrm{reg}}(M, L_{e,0}, L_{e,1})$ for all e, and that $J \in \mathscr{J}^{\mathrm{reg}}(E, \pi, Q, j, \{J_e\})$. Then $\operatorname{ind} D_{u,J} \geqslant 0$ and $\operatorname{ind} D_{\sigma_{e,m},J_e} > 0$ on the right-hand side of (2.25), so that the left-hand side can be zero only if $\operatorname{ind} D_{u,J} = 0$ and $l_e = 0$ for all e. It follows that the zero-dimensional part of $\mathscr{M}_J(\{x_e\})$, for any $\{x_e\}$, is compact, hence a finite set. Write $v_J(\{x_e\}) \in \mathbb{Z}/2$ for the number of points modulo two in this set, and consider the map

$$C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J) : \bigotimes_{e \in I^+} \mathrm{CF}(L_{e,0}, L_{e,1}) \to \bigotimes_{e \in I^-} \mathrm{CF}(L_{e,0}, L_{e,1}) \tag{2.26}$$

given by the matrix with entries $v_J(\{x_e\})$, that is to say

$$C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J) \left(\underset{e \in I^+}{\otimes} \langle x_e \rangle \right) = \sum_{\{x_e\}_{e \in I^-}} v_J(\{x_e\}_{e \in I^- \cup I^+}) \left(\underset{e \in I^-}{\otimes} \langle x_e \rangle \right).$$

A standard argument, involving the structure at infinity of the one-dimensional part of $\mathscr{M}_J(\{x_e\})$, shows is that there is an even number of points in $\bar{\mathscr{M}}_J(\{x_e\})$ of the following form: l_e is 1 for a single end $e = f$, and zero for all other e, so that the point is a pair $(u, \sigma_{f,1})$; and moreover $\operatorname{ind} D_{u,J} = 0$, $\operatorname{ind} D_{\sigma_{f,1},J_f} = 1$. Algebraically, what this says is that $C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J)$ is a chain map. The relative invariant $\Phi_0^{\mathrm{rel}}(E, \pi, Q)$ is defined to be the induced map on cohomology.

The next step is to show that this is independent of the choice of j and J, keeping the J_e fixed for the moment. Any two j^0, j^1 can be connected by a family j^μ, $0 \leqslant \mu \leqslant 1$, which remains constant on the ends; correspondingly, for $J^0 \in \mathscr{J}^{reg}(E, \pi, Q, j^0, \{J^e\})$, $J^1 \in \mathscr{J}^{\mathrm{reg}}(E, \pi, Q, j^1, \{J^e\})$ one can find a family J^μ joining them, which is regular in the parametrized sense (this is not the same as saying that each J^μ should itself be regular, which would be impossible to achieve in general). The parametrized moduli spaces

$$\mathscr{M}^{\mathrm{para}}_{(J^\mu)}(\{x_e\}) = \bigcup_{0 \leqslant \mu \leqslant 1} \{\mu\} \times \mathscr{M}_{J^\mu}(\{x_e\})$$

are then smooth manifolds, and they have a parametrized version of the Gromov–Floer compactification. In particular, the zero-dimensional parts are again finite sets, so that one can use the number of points $\lambda(\{x_e\}) \in \mathbb{Z}/2$ in them to define a map $h(E, \pi, Q, (J^\mu))$ between the same groups as in (2.26). Arguing along the same lines as when proving that $C\Phi_0^{\mathrm{rel}}$ is a chain map, one can show that h is a homotopy between $C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J^0)$ and $C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J^1)$. The same argument can be used to show that $\Phi_0^{\mathrm{rel}}(E, \pi, Q)$ remains invariant under deformations of the geometric data, that is to say of Q or (E, π) itself, as long as the structure of the ends remains unchanged. Finally, we should prove that the relative invariant is independent of the J_e, which more accurately means that it commutes with the canonical isomorphisms between Floer cohomology groups for different J_e. We omit this entirely, both because it is not important for our purpose, and because it would require a digression concerning "continuation maps".

Two ways of gluing together surfaces with strip-like ends will play a role later on. One of them is a close cousin of that considered as in Section 2.1. It can be formulated in various degrees of generality, but we will need only one special case. Suppose then that S^1 is a surface with strip-like ends, and S^2 a compact surface, together with points $\zeta^k \in \partial S^k$ (ζ^1 should not lie on any end; that can of course always be achieved by making the ends smaller). Let (E^k, π^k) be exact Lefschetz fibrations over S^k with Lagrangian boundary conditions Q^k; (E^1, π^1) should be trivial over the ends. We also want to have M, L, maps $\phi^k : M \to (E^k)_{\zeta^k}$, and trivializations (ψ^k, Ψ^k), with the same properties as in Section 2.1. The boundary connected sum $S = S^1 \#_{\zeta^1 \sim \zeta^2} S^2$ is a surface with the same kind of strip-like ends as S^1. As before one constructs an exact Lefschetz fibration (E, π) on it with a Lagrangian boundary condition Q, modelled over the ends on the same Lagrangian submanifolds as Q^1.

Choose complex structures j^k on S^k such that $(\psi^k)^* j^k$ is standard; j^1 should also be standard on the ends. Take $\{J_e\}$ and $J^1 \in \mathscr{J}^{\mathrm{reg}}(E^1, \pi^1, Q^1, j^1, \{J_e\})$ as when defining the relative invariant of (E^1, π^1, Q^1); as an additional condition, we want $(\Psi^1)^* J^1$ to be the product of the standard complex structure on $\bar{D}^+(1)$ and some fixed almost complex structure on M. By Lemma 2.20 this can be done while still achieving regularity. Using Lemmas 2.4 and 2.5 one finds a $J^2 \in \mathscr{J}^{\mathrm{reg}}(E^2, \pi^2, Q^2, j^2)$ with the same restriction on $(\Psi^2)^* J^2$, and such that the evaluations

$$\mathrm{ev}_{\zeta^1} \mid \mathscr{M}_{J^1}(\{x_e\}) : \mathscr{M}_{J^1}(\{x_e\}) \to Q^1_{\zeta^1} \cong L,$$

$$\mathrm{ev}_{\zeta^2} \mid \mathscr{M}_{J^2} : \mathscr{M}_{J^2} \to Q^2_{\zeta^2} \cong L \tag{2.27}$$

are transverse to each other for every $\{x_e\}$. Let $J \in \mathscr{J}(E, \pi, j, \{J_e\})$ be the almost complex structure constructed from J^1, J^2. As in Proposition 2.7, this will be regular for small values of the parameter ρ, and

$$\mathscr{M}_J(\{x_e\})_{[0]} \cong (\mathscr{M}_{J^1}(\{x_e\}) \times_L \mathscr{M}_{J^2})_{[0]}, \tag{2.28}$$

where the [0] denotes on both sides the zero-dimensional component of these manifolds. We will not need the full "gluing formula" which one can obtain from this, but only a special case:

Lemma 2.21. *If* $\Phi_1(E^2, \pi^2, Q^2) = 0$ *then* $\Phi_0^{\mathrm{rel}}(E, \pi, Q) = 0$.

Proof. For simplicity, suppose that $\Phi_1(E^2, \pi^2, Q^2)$ vanishes even when taken in the cobordism ring $MO_*(L)$. This means that there is a compact manifold with boundary G and a smooth map $g : G \to L$, such that $\partial G = \mathscr{M}_{J^2}$ and $g | \partial G = ev_{\zeta^2}$. After perturbing it slightly, one can assume that g is transverse to all maps ev_{ζ^1} in (2.27). Then the fibre products

$$\mathscr{G}(\{x_e\}) = \mathscr{M}_{J^1}(\{x_e\}) \times_L G \tag{2.29}$$

are smooth manifolds. The evaluation map extends continuously to the Gromov–Floer compactification, so that one can define compactifications $\bar{\mathscr{G}}(\{x_e\})$ in the obvious way. It is not difficult to see, using (2.26), that the zero-dimensional part $\mathscr{G}(\{x_e\})_{[0]}$ is a finite set; one counts the points

$\xi_{J,G,g}(\{x_e\}) \in \mathbb{Z}/2$ in it, and uses that to define a map k between the Floer cochain groups associated to (E^1, π^1, Q^1), as in (2.26). We claim that this is a homotopy between $C\Phi^0_{\mathrm{rel}}(E, \pi, Q, J)$ and the zero map. As usual, the proof is based on analysing the ends of the one-dimensional moduli spaces. The closure in $\bar{\mathscr{G}}(\{x_e\})$ of the one-dimensional part $\mathscr{G}(\{x_e\})_{[1]}$ is a compact one-manifold with boundary, and its boundary points are of two kinds: first, boundary points of $\mathscr{G}(\{x_e\})_{[1]}$ itself, $\partial\mathscr{G}(\{x_e\})_{[1]} = (\mathscr{M}_{J^1}(\{x_e\}) \times_L \partial G)_{[0]} \cong \mathscr{M}_J(\{x_e\})_{[0]}$; their number modulo two is $v_J(\{x_e\})$ by definition. The second kind of boundary points are of the form $(u, \sigma_{f,1}) \times q \in \bar{\mathscr{M}}_J(\{x_e\}) \times_L G$. This means that for some end f, and points $\{\hat{x}_e\}$ with $\hat{x}_e = x_e$ for all $e \neq f$, one has

$$u \times q \in \mathscr{G}(\{\hat{x}_e\})_{[0]}, \qquad \sigma_{f,1} \in \begin{cases} (\mathscr{F}^*(x_f, \hat{x}_f)/\mathbb{R})_{[0]} & \text{if } e \text{ is a negative end,} \\ (\mathscr{F}^*(\hat{x}_f, x_f)/\mathbb{R})_{[0]} & \text{if } e \text{ is a positive end.} \end{cases}$$

The number of such boundary points is

$$\sum_{f \in I^-} \sum_{\hat{x}_f} n_{J_f}(x_f, \hat{x}_f) \xi_{J,G,g}(\{\hat{x}_e\}) + \sum_{f \in I^+} \sum_{\hat{x}_f} \xi_{J,G,g}(\{\hat{x}_e\}) n_{J_f}(\hat{x}_f, x_f) \in \mathbb{Z}/2.$$

The fact that this is equal to $v_J(\{x_e\})$ gives precisely the desired equality $dk + kd = C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J)$. □

The other and maybe more obvious gluing process is to join together two ends. Assume that S^1, S^2 are surfaces with strip-like ends, such that S^1 has a single positive end e^1, and S^2 a single negative end e^2 (this is not really a restriction since, as has been mentioned before, positive ends can be turned into negative ones and vice versa). Choose some $\sigma > 0$, and define $S = S^1 \#_{e^1 \sim e^2} S^2$ by taking $S^1 \setminus \gamma_{e^1}((\sigma; \infty) \times [0; 1])$ and $S^2 \setminus \gamma_{e^2}((-\infty; -\sigma) \times [0; 1])$, and identifying $\gamma_{e^1}(s, t)$ with $\gamma_{e^2}(s - \sigma, t)$ for $(s, t) \in [0; \sigma] \times [0; 1]$. The ends of S are the negative ends of S^1 together with the positive ends of S^2. After choosing complex structures j^k on S^k which are standard over the ends, there is an obvious induced complex structure j on S.

Let (L_0, L_1) be a pair of exact Lagrangian submanifolds in M. Suppose that we have exact Lefschetz fibrations (E^k, π^k) over S^k, trivial over the ends, and Lagrangian boundary conditions Q^1, Q^2 for them modelled on (L_0, L_1) over e^1 and over e^2, respectively. Then one can form an exact Lefschetz fibration (E, π) over S by identifying $\Gamma_{e^1}(s, t, y) \in E^1$ with $\Gamma_{e^2}(s - \sigma, t, y) \in E^2$, in parallel with the construction on the base. This comes with an obvious Lagrangian boundary condition Q. Choose almost complex structures J^k on E^k so as to define relative invariants $C\Phi_0^{\mathrm{rel}}(E^k, \pi^k, Q^k, J^k)$. We require that the $J_{e^k} \in \mathscr{J}^{\mathrm{reg}}(M, L_0, L_1)$ on which J^k is modelled over the end e^k should be the same for $k = 1, 2$. Then J^1 and J^2 match up to an almost complex structure J on E, which is compatible relative to j.

Proposition 2.2. *For fixed J^1, J^2, if one chooses σ to be sufficiently large, then J is regular. Moreover, denoting again by* [0] *the zero-dimensional components, and by I^-, I^+ the negative and positive ends of S, one has*

$$\mathscr{M}_J(\{x_e\})_{[0]} \cong \bigcup_{x \in L_0 \cap L_1} \mathscr{M}_{J^1}(\{x_e\}_{e \in I^-}, x)_{[0]} \times \mathscr{M}_{J^2}(x, \{x_e\}_{e \in I^+})_{[0]}$$

for any $\{x_e\}_{e \in I^- \cup I^+}$.

The method used in the proof of this is the same as when setting up Floer cohomology. A thorough exposition of a closely related result can be found in [24, Section 4.4]. The implication for the relative invariants is clear:

$$C\Phi_0^{\mathrm{rel}}(E,\pi,Q,J) = C\Phi_0^{\mathrm{rel}}(E^1,\pi^1,Q^1,J^1) \circ C\Phi_0^{\mathrm{rel}}(E^2,\pi^2,Q^2,J^2), \tag{2.30}$$

which proves the main TQFT-style property of relative invariants, namely, that they are functorial if one regards the gluing $S = S^1 \#_{e^1 \sim e^2} S^2$ as composition of the surfaces S^1, S^2.

2.5. Horizontality and relative invariants

The aim of this section is to extend the methods of Section 2.2 to surfaces with strip-like ends. Throughout, (E,π) will be an exact Lefschetz fibration over such a surface S, trivial on the ends, together with a Lagrangian boundary condition Q modelled over the ends on pairs $L_{0,e}, L_{1,e} \subset M$ of transversally intersecting exact Lagrangian submanifolds. For $\{x_e\}_{e \in I^- \cup I^+}$ with $x_e \in L_{0,e} \cap L_{1,e}$ write

$$\chi(\{x_e\}) = \sum_{e \in I^-} a_{L_{0,e},L_{1,e}}(x_e) - \sum_{e \in I^+} a_{L_{0,e},L_{1,e}}(x_e).$$

By (2.23) $A(u) = \chi(\{x_e\}) + \int_{\partial S} \kappa_Q$ for all $u \in \mathscr{B}(\{x_e\})$. After fixing a complex structure j on S which is standard on the ends, and a $J_e \in \mathscr{J}(M)$ for each e, we consider the space

$$\mathscr{J}^{\mathrm{h}}(E,\pi,j,\{J_e\}) = \mathscr{J}^{\mathrm{h}}(E,\pi,j) \cap \mathscr{J}(E,\pi,j,\{J_e\})$$

of almost complex structures J, compatible relative to j, which are horizontal and have prescribed behaviour on the ends. The two conditions do not contradict each other, since the model on each end, the almost complex structure $j_\pm \times J_e$ on the trivial fibration $\mathbb{R}^\pm \times [0;1] \times M \to \mathbb{R}^\pm \times [0;1]$, is obviously horizontal. A little more thought shows that $\mathscr{J}^{\mathrm{h}}(E,\pi,j,\{J_e\})$ is contractible. Choosing J in that space, one finds that for all $u \in \mathscr{B}(\{x_e\})$,

$$\tfrac{1}{2}\int_S \|(Du)^{\mathrm{v}}\|^2 + \int_S f(u)\beta = A(u) + \int_S \|\bar{\partial}_J u\|^2. \tag{2.31}$$

Here $\|\cdot\|$ is the metric on TE^{v} associated to Ω and J; $\beta \in \Omega^2(S)$ is a positive two-form; and f is the function defined by $\Omega|TE^{\mathrm{h}} = f(\pi^*\beta|TE^{\mathrm{h}})$. Because the symplectic connection is trivial on the ends, $f(u)$ is compactly supported. Eq. (2.31) is obviously the analogue of (2.10), and can be proved in the same way. Recall that (E,π) has nonnegative curvature iff $f \geqslant 0$. From this we draw two conclusions:

Lemma 2.23. *Assume that (E,π) has nonnegative curvature, and choose $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j,\{J_e\})$. Then $\mathscr{M}_J(\{x_e\}) = \emptyset$ for all $\{x_e\}$ with $\chi(\{x_e\}) < -\int_{\partial S} \kappa_Q$.*

Lemma 2.24. *In the same situation, let α be the minimum of $a_{L_{0,e},L_{1,e}}(x_-) - a_{L_{0,e},L_{1,e}}(x_+)$, taken over all e and all $x_\pm \in L_{0,e} \cap L_{1,e}$ for which this is > 0. Then the space $\mathscr{M}_J(\{x_e\})$ is compact, in the Gromov–Floer topology, for all $\{x_e\}$ such that $\chi(\{x_e\}) < -\int_{\partial S} \kappa_Q + \alpha$.*

Proof. Consider a sequence (u_i) in $\mathscr{M}_J(\{x_e\})$ which converges to $(u, \{\sigma_{e,m}\}) \in \bar{\mathscr{M}}_J(\{x_e\})$. The definition of γ implies that in (2.24), each $a_{L_{e,0},L_{e,1}}(\hat{x}_{e,m-1}) - a_{L_{e,0},L_{e,1}}(\hat{x}_{e,m}) \geqslant \alpha$; therefore

$$A(u) \leqslant A(u_i) - \left(\sum_e l_e\right)\alpha.$$

By assumption $A(u_i) = \chi(\{x_e\}) + \int_{\partial S} \kappa_Q < \alpha$, while on the other hand, $A(u) \geqslant 0$ by nonnegative curvature and (2.31). This is only possible if $l_e = 0$ for all e, so that the limit actually lies in $\mathscr{M}_J(\{x_e\})$. □

As before, denote by $\mathscr{M}^{\mathrm{h}}$ the space of horizontal sections with boundary values in Q. The presence of strip-like ends makes this space somewhat simpler than in the case which we have encountered before. Take an arbitrary $u \in \mathscr{B}$ and consider $u_e(s,t) = \Gamma_e^{-1} \circ u \circ \gamma_e(s,t) = (s, t, \sigma_e(s,t))$; u is horizontal over the end e iff $\sigma_e \equiv x \in M$ is constant, in which case the boundary conditions $\sigma_e(s,0) \in L_{e,0}$, $\sigma_e(s,1) \in L_{e,1}$ imply that $x \in L_{e,0} \cap L_{e,1}$. Since a horizontal section is determined by its value at any one point, for any e and any $x \in L_{e,0} \cap L_{e,1}$ there can be at most one $u \in \mathscr{M}^{\mathrm{h}}$ such that $\sigma_e \equiv x$. Thus $\mathscr{M}^{\mathrm{h}}$ is a finite disjoint union of the subsets $\mathscr{M}^{\mathrm{h}}(\{x_e\}) = \mathscr{M}^{\mathrm{h}} \cap \mathscr{B}(\{x_e\})$, each of which consists of at most one element. One easily proves the following limiting case of Lemma 2.23:

Lemma 2.25. *For (E,π) and J as in Lemma 2.23, suppose that $\{x_e\}$ satisfy $\chi(\{x_e\}) = -\int_{\partial S} \kappa_Q$; then $\mathscr{M}_J(\{x_e\}) = \mathscr{M}^{\mathrm{h}}(\{x_e\})$.*

To translate these elementary observations into results about relative invariants, one needs to address again the question of regularity of horizontal almost complex structures; in other words, what is required are analogues of Lemmas 2.9 and 2.11. For the first of these, both the statement and proof as essentially the same as in the original situation; the second needs to be adapted a little.

Lemma 2.26. *Let $U \subset S$ be a nonempty open subset disjoint from the ends, such that any partial section $w: U \to E|U$ which is horizontal and satisfies $w(\partial S \cap U) \subset Q$ is the restriction of a $u \in \mathscr{M}^{\mathrm{h}}$. Then, given some $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j,\{J_e\})$, there are $J' \in \mathscr{J}^{\mathrm{h}}(E,\pi,j,\{J_e\})$ arbitrarily close to it and which agree with it outside $\pi^{-1}(U)$, such that for all $\{x_e\}$, any $u \in \mathscr{M}_J(\{x_e\}) \setminus \mathscr{M}^{\mathrm{h}}$ is regular.*

Lemma 2.27. *Let (E,π) and J be as in Lemma 2.23. If $u \in \mathscr{M}^{\mathrm{h}}$ is a horizontal section with $A(u)=0$, then* $\ker D_{u,J} = 0$.

Proof. Formally, taking the second derivative of (2.31) at u yields the same formula (2.15) as in the case of a compact S, but a little care needs to be exercised about its validity. It certainly holds for those elements of $T\mathscr{B}_u = \{X \in C^\infty(u^*TE^{\mathrm{v}})\colon X_z \in T(Q_z) \text{ for } z \in \partial S\}$ which are compactly supported, and by continuity, for all X in the $W^{1,2}$-completion; to be precise, this Sobolev space is with respect to the metric $\|\cdot\|$ and the connection ∇^u on u^*TE^{v}. We actually want to use the formula with $X \in \mathscr{W}_u^1$ and this is a space of $W^{1,p}$-sections with $p > 2$, hence not contained in $W^{1,2}$. However, if one assumes additionally that $D_{u,J}X = 0$ there is no problem, because any such X is smooth and decays exponentially on the ends, as do its derivatives.

From $A(u) = 0$ and (2.31) it follows that $f(u) \equiv 0$, so that the Hessians $\mathrm{Hess}(f|E_z)_{u(z)}$ are nonnegative. One then sees from (2.15) that any $X \in \mathscr{W}^1_u$ with $D_{u,J}X = 0$ satisfies $\nabla^u X = 0$. Choose some end e and write $x = x_e$. The trivialization Γ_e induces a trivialization of the vector bundle $\gamma_e^*(u^*TE^{\mathrm{v}}) \to \mathbb{R}^{\pm} \times [0;1]$, which identifies it with the trivial bundle with fibre TM_x. In this way, $Y = \gamma_e^* X$ becomes a map $\mathbb{R}^{\pm} \times [0;1] \to TM_x$. Since the connection ∇^u is compatible with the trivialization, $\nabla^u X = 0$ implies that Y must be constant. On the other hand, the boundary conditions which are part of the definition of $T\mathscr{B}_u$ and of $\mathscr{W}^1_u$ tell us that $Y_{s,k} \in T(L_{e,k})_x$ for $k = 0, 1$. Because the $L_{e,k}$ intersect transversally, it follows that $Y = 0$, hence that $X|\mathrm{im}(\gamma^e) = 0$. Since X is covariantly constant, it must be zero everywhere. $\square$

The next result summarizes what progress we have made so far, as well as the implications for the coefficients $v_J(\{x_e\})$ of the relative invariant.

Proposition 2.28. *Assume that (E,π) has nonnegative curvature, and that any $u \in \mathscr{M}^{\mathrm{h}}$ satisfies $A(u) = 0$ and $\mathrm{ind}\, D_{u,J} = 0$. Set $\kappa = \int_{\partial S} \kappa_Q$.*

(i) *Let $U \subset S$ be a nonempty open subset, disjoint from the ends, such that any partial horizontal section $w : U \to E|U$ with $w(\partial S \cap U) \subset Q$ extends to a $u \in \mathscr{M}^{\mathrm{h}}$. Then for any $J \in \mathscr{J}^{\mathrm{h}}(E,\pi,j,\{J_e\})$ there are $J' \in \mathscr{J}^{\mathrm{reg,h}}(E,\pi,Q,j,\{J_e\}) = \mathscr{J}^{\mathrm{h}}(E,\pi,j,\{J_e\}) \cap \mathscr{J}^{\mathrm{reg}}(E,\pi,Q,j,\{J_e\})$* arbitrarily close to J, and which agree with it outside $\pi^{-1}(U)$.

(ii) *If $J_e \in \mathscr{J}^{\mathrm{reg}}(M,L_{0,e},L_{1,e})$ for all e, and $J \in \mathscr{J}^{\mathrm{reg,h}}(E,\pi,Q,j,\{J_e\})$, then*

$$v_J(\{x_e\}) = \begin{cases} 0 & \text{if } \chi(\{x_e\}) < -\kappa, \\ \#\mathscr{M}^{\mathrm{h}}(\{x_e\}) & \text{if } \chi(\{x_e\}) = -\kappa. \end{cases}$$

Proof. (i) Take a J' as given by Lemma 2.26. Then all $u \in \mathscr{M}_{J'}(\{x_e\})$ are regular except possibly for the horizontal ones. These satisfy $A(u) = 0$, so we can apply Lemma 2.27 to them, showing that $\ker D_{u,J'} = 0$, and by assumption on the index, that $\mathrm{coker}\, D_{u,J'} = 0$; which means that they are regular as well.

(ii) This follows immediately from Lemmas 2.23 and 2.25, in view of the definition of $v_J(\{x_e\})$. $\square$

An algebraic language suitable for encoding results of this kind is that of $\mathbb{R}$-graded vector spaces, that is to say vector spaces C equipped with a splitting $C = \oplus_{r \in \mathbb{R}} C_r$. All vector spaces occurring here will be over $\mathbb{Z}/2$ and finite-dimensional; in particular, their support $\mathrm{supp}(C) = \{r \in \mathbb{R} : C_r \neq 0\}$ is always a finite set. Let $I \subset \mathbb{R}$ be an interval. We say that C has *gap I* if there are no $r, s \in \mathrm{supp}(C)$ with $r - s \in I$. A map $f : C \to D$ between graded $\mathbb{R}$-vector spaces is said to be *of order I* if $f(C_r) \subset \oplus_{s \in r+I} D_s$ for all r.

Floer cochain groups are obvious examples: $C = \mathrm{CF}(L_0, L_1)$ is canonically $\mathbb{R}$-graded, with C_r the subspace spanned by those $\langle x \rangle$ with $a_{L_0,L_1}(x) = r$. Because of its gradient flow interpretation, the Floer differential d_J is of order $(0;\infty)$. In a rather trivial way, this can always be strengthened slightly; namely, there is an $\alpha > 0$ such that C has gap $(0;\alpha)$, and then d_J is of order $[\alpha;\infty)$. One can reformulate Proposition 2.28(ii) in this language as follows:

Lemma 2.29. *Take (E,π), Q and κ as in Proposition* 2.28, *with J_e, J as in part* (ii). *Then the map $C\Phi^0_{\mathrm{rel}}(E,\pi,Q,J)$* is of order $[-\kappa;\infty)$. *For a more precise statement*, *take $\alpha > 0$ to be the minimum of $\chi(\{x_e\}) + \kappa$, ranging over all $\{x_e\}$ where this is positive. Then $C\Phi_0^{\mathrm{rel}}(E,\pi,Q,J) = \phi + (C\Phi_0^{\mathrm{rel}}(E,\pi,Q,J) - \phi)$, where the first summand is of order $\{-\kappa\}$ and the second of order $[-\kappa+\alpha;\infty)$. Moreover, ϕ is determined by the horizontal sections*:

$$\phi\left(\underset{e\in I^+}{\otimes}\langle x_e\rangle\right) = \sum_{\{x_e\}_{e\in I^-}} \#\mathscr{M}^{\mathrm{h}}(\{x_e\}_{e\in I^-\cup I^+})\left(\underset{e\in I^-}{\otimes}\langle x_e\rangle\right).$$

There is a version of this for the homotopies between the $C\Phi^0_{\mathrm{rel}}$ for different choices of (j,J). The proof applies the same ideas as before to parametrized moduli spaces, and is left to the reader.

Lemma 2.30. *Let (E,π), Q and κ be as in Proposition* 2.28. *Consider two complex structures j^k on S, $k=0,1$, and correspondingly two almost complex structures $J^k \in \mathscr{J}^{\mathrm{reg,h}}(E,\pi,Q,j,\{J_e\})$; note that the J_e are supposed to be the same for both k. Then the maps $C\Phi^0_{\mathrm{rel}}(E,\pi,Q,J^k)$ are homotopic by a chain homotopy which is of order $(-\kappa;\infty)$.*

It is a familiar idea that when dealing with maps between $\mathbb{R}$-graded vector spaces, the "lowest order term" is usually the most important, and knowing it is often sufficient to resolve a question. A particular instance of this is relevant for our purpose.

Lemma 2.31. *Let D be an $\mathbb{R}$-graded vector space with a differential d_D of order* $[0;\infty)$. *Suppose that D has gap $[\varepsilon;2\varepsilon)$ for some $\varepsilon > 0$. One can then write $d_D = \delta + (d_D - \delta)$ with δ of order* $[0;\varepsilon)$, *satisfying $\delta^2 = 0$, and $(d_D - \delta)$ of order $[2\varepsilon;\infty)$. Suppose that in addition, $H(D,\delta) = 0$; then $H(D,d_D) = 0$.*

Proof. Thanks to the gap assumption, $\mathrm{supp}(D)$ can be decomposed into disjoint subsets $R_1,\ldots,R_m$ such that for $r\in R_i$, $s\in R_j$,

$$r-s\begin{cases}\leqslant -2\varepsilon, & i<j,\\ \in(-\varepsilon;\varepsilon) & i=j,\\ \geqslant 2\varepsilon & i>j.\end{cases}$$

Define a descending filtration of (D,d_D),

$$F^k = \bigoplus_{r\in R_k\cup R_{k+1}\cup\cdots\cup R_m} D_r.$$

There is a "spectral sequence" which takes the form of a sequence (E^k,∂^k) of differential vector spaces, such that $E^{k+1} = H(E^k,\partial^k)$. It starts with $E^0 = \oplus_i F^i/F^{i+1}$, which has a differential ∂^0 induced by d_D. In our case this can be identified with (D,δ), so the assumption says that $E^1 = 0$. On the other hand, $E^k \cong H(D,d_D)$ for $k \gg 0$. □

Lemma 2.32. *Take three $\mathbb{R}$-graded vector spaces C', C, C'', each of them with a differential of order $(0;\infty)$. Suppose that we have differential maps $b: C' \to C$, $c: C \to C''$ and a homotopy*

$h: C' \to C''$ between $c \circ b$ and the zero map, such that the following conditions are satisfied for some $\varepsilon > 0$:

(i) *C', C'' have gap $(0; 3\varepsilon)$, and C has gap $(0; 2\varepsilon)$.*
(ii) *For all $r \in \mathrm{supp}(C')$ and $s \in \mathrm{supp}(C'')$, $|r - s| \geqslant 4\varepsilon$.*
(iii) *One can write $b = \beta + (b - \beta)$ with β of order $[0; \varepsilon)$ and $(b - \beta)$ of order $[2\varepsilon; \infty)$; and $c = \gamma + (c - \gamma)$ with the same properties. The low order parts (which do not need be differential maps) fit into a short exact sequence of vector spaces*

$$0 \to C' \xrightarrow{\beta} C \xrightarrow{\gamma} C'' \to 0.$$

(iv) *h is of order $[0; \infty)$.*

Then the maps on cohomology induced by b, c fit into a long exact sequence

$$\begin{tikzcd} H(C', d_{C'}) \arrow[r, "b_*"] & H(C, d_C) \arrow[r, "c_*"] & H(C'', d_{C''}) \arrow[ll, bend left] \end{tikzcd}. \tag{2.32}$$

Proof. Consider intervals $I_r = [r; r + \varepsilon)$ for $r \in \mathrm{supp}(C')$, and $I_r = (r - \varepsilon; r]$ for $r \in \mathrm{supp}(C'')$. By (i), (ii) these are pairwise disjoint, and the distance between any two of them is $\geqslant 2\varepsilon$. From (iii) one sees that $\mathrm{supp}(C)$ is contained in the union of these intervals, which shows that $D = C' \oplus C \oplus C''$ has gap $[\varepsilon; 2\varepsilon)$. Consider the differential $d_D = \delta + (d_D - \delta)$,

$$d_D = \begin{pmatrix} d_{C'} & 0 & 0 \\ b & d_C & 0 \\ h & c & d_{C''} \end{pmatrix}, \qquad \delta = \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix}.$$

We know that $d_{C'}$, d_C, $d_{C''}$, $(b - \beta)$, $(c - \gamma)$ are of order $[2\varepsilon; \infty)$. Combining (ii) with (iv) shows that h is of order $[4\varepsilon; \infty)$, so that $(d_D - \delta)$ is of order $[2\varepsilon; \infty)$. On the other hand, δ is of order $[0; \varepsilon)$, and (iii) says that $H(D, \delta) = 0$. Lemma 2.31 shows that $H(D, d_D) = 0$, which by a general fact implies the existence of a long exact sequence (2.32). $\square$

3. Wrapping it up

Technically, the proof of the exact sequence is an application of Lemma 2.32. The spaces C', C, C'' which occur in the lemma will be Floer cochain spaces, and the maps b, c relative invariants. While the definition of b is rather straightforward, that of c uses some of the geometry from Chapter 1, specifically the standard fibrations constructed in Section 1.2. In both cases, the desired properties follow from nonnegative curvature, that is to say Proposition 2.28 and related results. The homotopy h is obtained by comparing two different constructions of a particular exact Lefschetz fibration. Nonnegative curvature again plays a role in analysing it, but the vanishing theorem of Section 2.3 is also important.

3.1. Preliminaries

This section sets up the framework for the whole chapter. The data are: M is an exact symplectic manifold. $L_0, L_1 \subset M$ are exact Lagrangian submanifolds. $L \subset M$ is an exact Lagrangian sphere, which comes with a diffeomorphism $f : S^n \to L$ and a symplectic embedding $\imath : T(\lambda) \to M$ for some $\lambda > 0$, such that $\imath | T(0) = f$. We use the Dehn twist τ_L defined using $\imath$ and a function R. Also given are small constants $\varepsilon, \delta > 0$. The following conditions are required to hold:

(I) $L \cap L_0$, $L \cap L_1$, $L_0 \cap L_1$ are transverse intersections, and $L \cap L_0 \cap L_1 = \emptyset$.

(II) The actions $a_{L_0,L_1}(x)$ of distinct points $x \in L_0 \cap L_1$ differ by at least 3ε. Secondly, as (x_0, x_1) runs over $(L_0 \cap L) \times (L \cap L_1)$, the numbers $a_{L_0,L}(x_0) + a_{L,L_1}(x_1)$ differ pairwise by at least 3ε. Thirdly, for any $x \in L_0 \cap L_1$, $(x_0, x_1) \in (L_0 \cap L) \times (L \cap L_1)$ one has

$$|a_{L_0,L_1}(x) - a_{L_0,L}(x_0) - a_{L,L_1}(x_1)| \geqslant 5\varepsilon.$$

(III) For all $y_k \in f^{-1}(L \cap L_k)$, $k = 0, 1$, the distance $\mathrm{dist}(y_0, y_1)$ in the standard metric on S^n is $\geqslant 2\pi\delta$.

(IV) $\imath^*\theta = \theta_T | T(\lambda)$ is the standard one-form, and the function K_L associated to L is zero. Moreover, each $\imath^{-1}(L_k) \subset T(\lambda)$ is a union of fibres; one can write this as

$$\imath^{-1}(L_k) = \bigcup_{y \in \imath^{-1}(L \cap L_k)} T(\lambda)_y. \tag{3.1}$$

(V) R satisfies $0 \geqslant 2\pi R(0) > -\varepsilon$, and is such that τ_L is δ-wobbly.

Remark 3.1. Since we will establish the exact sequence under these conditions, it is necessary to convince ourselves that they do not restrict its validity in any way. Suppose then that we are given arbitrary exact Lagrangian submanifolds L_0, L_1 and a framed exact Lagrangian sphere $(L, [f])$ in M. After perturbing the submanifolds slightly, one can assume that (I) holds. Another such perturbation achieves (II) for some $\varepsilon > 0$. This is an instance of a general fact: by moving one of two transverse exact Lagrangian submanifolds slightly, the action of the intersection points can be changed independently of each other, by arbitrary sufficiently small amounts. Choose some representative f of the framing. Since $L_0 \cap L_1 \cap L = \emptyset$, (III) is automatically true for some $\delta > 0$. Because the intersections $L \cap L_k$ are transverse, one can find a symplectic embedding $\imath : T(\lambda) \to M$, for some $\lambda > 0$, which extends f and such that $\imath^{-1}(L \cap L_k)$ is a union of fibres. By replacing θ with $\theta + \mathrm{d}H$ for a suitable H, and making λ smaller, one can ensure that $\imath^*\theta = \theta_T$ is satisfied. Note that when one modifies θ in this way, the functions associated to exact Lagrangian submanifolds change accordingly. One can use this and the freedom in the choice of H to arrange that K_L becomes equal to zero. In any case, the values of the action functional at intersection points remain the same, so that this does not interfere with (II). We have now satisfied (IV). It is no problem to choose R such that (V) holds for the previously obtained ε, δ. None of the changes which we have made affects Floer cohomology. Therefore, once the exact sequence is established for the modified data, it also holds for the original ones.

Next, we need to draw some elementary inferences. Condition (IV) implies that $\mathrm{d}K_{L_k} = \theta | L_k$ vanishes on $L_k \cap \mathrm{im}(\imath)$; in other words $K_{L_k} \circ \imath$ is constant on each fibre in (3.1). Let τ be the model

Dehn twist from which τ_L is constructed. The function K_τ associated to it was determined in Lemma 1.8. Now K_{τ_L} vanishes outside $\mathrm{im}(\imath)$ and, again by (IV), satisfies $K_{\tau_L} \circ \imath = K_\tau$. Concretely

$$K_{\tau_L}(\imath(y)) = 2\pi(R'(\|y\|)\|y\| - R(\|y\|))$$
$$= -2\pi R(0) + 2\pi \int_0^{\|y\|} (R'(\|y\|) - R'(t))\,\mathrm{d}t. \tag{3.2}$$

In particular $K_{\tau_L}|L = -2\pi R(0)$, which by (V) lies in $[0;\varepsilon)$. The same condition says that R' decreases monotonically from $R'(0) = \frac{1}{2}$ until it reaches the value δ, and thereafter takes values in $[0;\delta)$. By combining this with (3.2) one obtains the estimate, valid for all $y \in T(\lambda)$ with $R'(\|y\|) \geqslant \delta$,

$$-2\pi R(0) \geqslant K_{\tau_L}(\imath(y)) \geqslant -2\pi R(0) - 2\pi \int_0^\infty R'(t)\,\mathrm{d}t = 0. \tag{3.3}$$

Now consider the $\mathbb{R}$-graded vector spaces

$$C' = CF(L, L_1) \otimes CF(\tau_L(L_0), L),$$
$$C'' = CF(L_0, L_1).$$

The first part of (II) implies that C'' has gap $(0; 3\varepsilon)$. Clearly, a point $\tilde{x}_0$ lies in $\tau_L(L_0) \cap L$ iff $x_0 = \tau_L^{-1}(\tilde{x}_0)$ lies in $L_0 \cap L_1$. By definition of $K_{\tau_L(L_0)}$ and the computation above,

$$a_{\tau_L(L_0),L}(\tilde{x}_0) = a_{L_0,L}(x_0) - K_{\tau_L}(x_0) = a_{L_0,L}(x_0) + 2\pi R(0). \tag{3.4}$$

Hence C' can be identified with $\mathrm{CF}(L, L_1) \otimes \mathrm{CF}(L_0, L)$ up to a shift in the grading, which is by a constant of size $< \varepsilon$. It therefore follows from (II) that C' has gap $(0; 3\varepsilon)$, and that the distance between the supports of C', C'' is at least 4ε. To summarize, what we have shown is that C', C'' satisfy the assumptions (i), (ii) of Lemma 2.32.

Lemma 3.2. *$\tau_L(L_0)$, L_1 intersect transversally, and there are injective maps*

$$p : (\tau_L(L_0) \cap L) \times (L \cap L_1) \to \tau_L(L_0) \cap L_1,$$
$$q : L_0 \cap L_1 \to \tau_L(L_0) \cap L_1$$

such that $\tau_L(L_0) \cap L_1$ is the disjoint union of their images. These maps have the following properties:

(i) *q is the inclusion $q(x) = x$. It preserves the values of the action functional, $a_{\tau_L(L_0),L_1}(x) = a_{L_0,L_1}(x)$. Moreover, for any $w \in \tau_L(L_0) \cap L_1$ and $x \in L_0 \cap L_1$ with $w \neq q(x)$ one has $a_{L_0,L_1}(x) - a_{\tau_L(L_0),L_1}(w) \notin [0; 3\varepsilon)$.*

(ii) *Set $\tilde{x} = p(\tilde{x}_0, x_1)$. Then*

$$0 \leqslant a_{\tau_L(L_0),L_1}(\tilde{x}) - a_{\tau_L(L_0),L}(\tilde{x}_0) - a_{L,L_1}(x_1) < \varepsilon. \tag{3.5}$$

Moreover, for any $w \in \tau_L(L_0) \cap L_1$ and $(\tilde{x}_0, x_1) \in (\tau_L(L_0) \cap L) \times (L \cap L_1)$ with $w \neq p(\tilde{x}_0, x_1)$ one has $a_{\tau_L(L_0),L_1}(w) - a_{\tau_L(L_0),L}(\tilde{x}_0) - a_{L,L_1}(x_1) \notin [0; 3\varepsilon)$.

(iii) *Suppose that there are* $x_k \in L \cap L_k$, $k=0,1$, *whose preimages* $y_k = \imath^{-1}(x_k)$ *are antipodes on* S^n. *Since* $\tau|S^n$ *is the antipodal map*, $\tilde{x}_0 = \tau_L(x_0)$ *is equal to* x_1 (*hence* $x_1 \in \tau_L(L_0) \cap L \cap L_1$, *and these are all such triple intersection points*). *In that case* $p(\tilde{x}_0, x_1) = \tilde{x}_0 = x_1$, *and* $a_{\tau_L(L_0),L_1}(p(\tilde{x}_0,x_1)) = a_{\tau_L(L_0),L}(\tilde{x}_0) + a_{L,L_1}(x_1)$.

Proof. Conditions (I) and (IV) imply that $L_0 \cap L_1 \cap \mathrm{im}(\imath) = \emptyset$. Since τ_L is the identity outside $\mathrm{im}(\imath)$, one has $L_0 \cap L_1 = (\tau_L(L_0) \cap L_1) \setminus \mathrm{im}(\imath)$, so that q can indeed be defined to be the inclusion. The equality $a_{\tau_L(L_0),L_1}(x) = a_{L_0,L_1}(x)$ follows from the fact that K_{τ_L} vanishes outside $\mathrm{im}(\imath)$.

There is a bijective correspondence between pairs $(\tilde{x}_0, x_1) \in (\tau_L(L_0) \cap L) \times (L \cap L_1)$ and $(y_0, y_1) \in \imath^{-1}(L_0 \cap L) \times \imath^{-1}(L \cap L_1)$, given by setting $y_0 = \imath^{-1}(\tau_L^{-1}(\tilde{x}_0))$, $y_1 = \imath^{-1}(x_1)$. As a consequence of (IV),

$$\imath^{-1}(\tau_L(L_0) \cap L_1) = \bigcup_{y_0,y_1} \tau(T(\lambda)_{y_0}) \cap T(\lambda)_{y_1}. \tag{3.6}$$

Since τ is δ-wobbly (V) and $\mathrm{dist}(y_0,y_1) \geqslant 2\pi\delta$ (III), one can apply Lemma 1.9 which tells us that each subset on the right-hand side of (3.6) consists of exactly one point. Fix temporarily some (y_0,y_1) and write $\tau(T(\lambda)_{y_0}) \cap T(\lambda)_{y_1} = \{\tilde{y}\}$, $\tilde{x} = \imath(\tilde{y})$. One defines $p(\tilde{x}_0,x_1) = \tilde{x}$. Then

$$\begin{aligned} a_{\tau_L(L_0),L_1}(\tilde{x}) &= K_{L_1}(\tilde{x}) - K_{\tau_L(L_0)}(\tilde{x}) \\ &= K_{L_1}(\tilde{x}) - K_{L_0}(\tau_L^{-1}(\tilde{x})) - K_{\tau_L}(\tau_L^{-1}(\tilde{x})). \end{aligned}$$

By construction $\tilde{y}$ lies in the fibre $T(\lambda)_{y_1}$, and since $K_{L_1} \circ \imath$ is constant on fibres, $K_{L_1}(\tilde{x}) = K_{L_1}(x_1)$. The same reasoning shows that $K_{L_0}(\tau_L^{-1}(\tilde{x})) = K_{L_0}(x_0)$, where $x_0 = \tau_L^{-1}(\tilde{x}_0)$. Moreover, one sees from (3.2) that K_{τ_L} is invariant under τ_L, so that $K_{\tau_L}(\tau_L^{-1}(\tilde{x})) = K_{\tau_L}(\tilde{x})$. With this and (3.4) in mind, one continues the computation

$$\begin{aligned} a_{\tau_L(L_0),L_1}(\tilde{x}) &= K_{L_1}(x_1) - K_{L_0}(x_0) - K_{\tau_L}(\tilde{x}) \\ &= a_{L,L_1}(x_1) + a_{L_0,L}(x_0) - K_{\tau_L}(\tilde{x}) \\ &= a_{L,L_1}(x_1) + a_{\tau_L(L_0),L}(\tilde{x}_0) - K_{\tau_L}(\imath(\tilde{y})) - 2\pi R(0). \end{aligned} \tag{3.7}$$

Lemma 1.9 also says that $R'(\|\tilde{y}\|) \geqslant \delta$. Combining this with (3.3) and (V) shows that $-K_{\tau_L}(\imath(\tilde{y})) - 2\pi R(0)$ lies in $[0;\varepsilon)$, which completes our proof of (3.5).

It is clear from their definitions that p,q are injective. A point of $\tau_L(L_0) \cap L_1$ falls into $\mathrm{im}(q)$ or $\mathrm{im}(p)$ depending on whether it lies inside or outside $\mathrm{im}(\imath)$, hence the two images are disjoint and cover $\tau_L(L_0) \cap L_1$. The transversality follows from Lemma 1.9 for $\mathrm{im}(p)$ and from that of $L_0 \cap L_1$ for $\mathrm{im}(q)$. We now turn to the claim made in the last sentence of (i). Supposing that w is a point of $L_0 \cap L_1$ different from x, one has $|a_{\tau_L(L_0),L_1}(w) - a_{L_0,L_1}(x)| = |a_{L_0,L_1}(w) - a_{L_0,L_1}(x)| \geqslant 3\varepsilon$ by (II). In the remaining case, which is when $w = p(\tilde{x}_0,x_1)$, (3.5) shows that $|a_{\tau_L(L_0),L_1}(w) - a_{L_0,L_1}(x)| > |a_{\tau_L(L_0),L}(\tilde{x}_0) + a_{L,L_1}(x_1) - a_{L_0,L_1}(x)| - \varepsilon$. We already know that the supports of C', C'' are at least 4ε apart, and one concludes that $|a_{\tau_L(L_0),L_1}(w) - a_{L_0,L_1}(x)| > 3\varepsilon$. A similar argument, paying a little more attention to signs, proves the parallel statement in (ii). Finally, the only nonobvious things in (iii) are the fact that $p(\tilde{x}_0,x_1) = x_1$ and the statement about the action. But these follow from Lemma 1.9 and the definition of p, respectively, from (3.7) and $K_{\tau_L}|L = -2\pi R(0)$. □

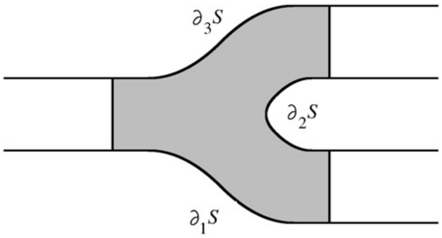

Fig. 3.

Lemma 3.2 and (II) imply that the actions $a_{\tau_L(L_0),L_1}(x)$ of different points $x \in \tau_L(L_0) \cap L_1$ differ by at least 2ε. In fact, for two such points which lie in $\mathrm{im}(q)$ the statement follows from (i) in the lemma; for two points which lie in $\mathrm{im}(p)$, from (ii); and combining the two parts shows it when one point lies in $\mathrm{im}(p)$ and the other in $\mathrm{im}(q)$. Set

$$C = CF(\tau_L(L_0), L_1).$$

We have just seen that this has gap $(0; 2\varepsilon)$. Define maps $\beta : C' \to C$, $\gamma : C \to C''$ by $\beta(\langle x_1 \rangle \otimes \langle \tilde{x}_0 \rangle) = \langle p(\tilde{x}_0, x_1) \rangle$ and $\gamma(\langle p(\tilde{x}_0, x_1) \rangle) = 0, \gamma(\langle q(x) \rangle) = \langle x \rangle$. The result above shows that these are of order $[0; \varepsilon)$ and fit into a short exact sequence as in Lemma 2.32(iii). What remains to be done is to realize them as "low-order parts" of chain maps b, c, and to construct the homotopy h.

3.2. The first map

Let S be the surface in Fig. 3, which has three boundary components $\partial_k S$ and three strip-like ends (two positive and a negative one). Take the trivial exact symplectic fibration π: $E = S \times M \to S$, with $\Omega \in \Omega^2(E)$, $\Theta \in \Omega^1(E)$ pulled back from ω, θ. Equip this with the Lagrangian boundary condition $Q = (\partial_1 S \times \tau_L(L_0)) \cup (\partial_2 S \times L) \cup (\partial_3 S \times L_1)$; κ_Q is zero, and $K_Q(z,x)$ is equal to $K_{\tau_L(L_0)}(x)$, $K_L(x)$ or $K_{L_1}(x)$, for z in the respective component $\partial_k S$. This gives rise to a relative invariant

$$\Phi_0^{\mathrm{rel}}(E, \pi, Q)\colon\ \mathrm{HF}(L, L_1) \otimes \mathrm{HF}(\tau_L(L_0), L) \to \mathrm{HF}(\tau_L(L_0), L_1).$$

The purpose of this section is to analyse this more closely, on the cochain level. Fix a complex structure j on S, trivial over the ends, and let $U \subset S$ be the open set shaded in Fig. 3.

Lemma 3.3. (E, π, Q) *and* U *satisfy the conditions of Proposition* 2.28(i).

Proof. The curvature of (E, π) is zero. A section $u(z) = (z, \sigma(z))$ is horizontal iff $\sigma(z) \equiv x \in M$ is constant, and it further satisfies $u(\partial S) \subset Q$ iff $x \in \tau_L(L_0) \cap L \cap L_1$. If $W \subset S$ is a connected open subset which intersects all three boundary components, the same description applies to partial horizontal section $w : W \to E|W$ with $w(W \cap \partial S) \subset Q$. As a consequence, any such section can be extended to $u \in \mathscr{M}^{\mathrm{h}}$. This is in particular true for $W = U$.

Fix some $x \in \tau_L(L_0) \cap L \cap L_1$ and the corresponding constant section $u \in \mathscr{M}^{\mathrm{h}}$. By definition $A(u) = 0$, and it remains to prove that $\mathrm{ind}\, D_{u,J} = 0$. From the description of the points x in Lemma 3.2(iii), together with the corresponding local statement in Lemma 1.9, it follows that there is a symplectic isomorphism $TM_x \cong \mathbb{C}^n$ which takes the tangent spaces to $\tau_L(L_0)$, L, and L_1 to $\mathbb{R}^n$, $\mathrm{e}^{2\pi \mathrm{i}/3}\mathbb{R}^n$, and

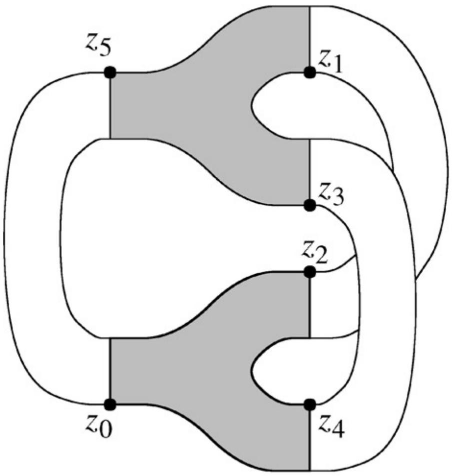

Fig. 4.

$e^{\pi i/3}\mathbb{R}^n$, respectively. Since the index is independent of the almost complex structure, we may choose $J = j \times J^M$ to be the product of j and some ω-compatible J^M on M. We may also assume that the isomorphism $TM_x \cong \mathbb{C}^n$ takes J_x^M to the standard complex structure. By definition, the domain of $D_{u,J}$ are sections of the vector bundle $u^*(TE^{\mathrm{v}}, J|TE^{\mathrm{v}}) \cong S \times TM_x$, with boundary conditions given by the tangent spaces to $\tau_L(L_0)$, L, L_1. Its range are $(0,1)$-forms with values in the same vector bundle. The preceding discussion allows us to identify

$$\mathscr{W}_u^1 = \{X \in W^{1,p}(S, \mathbb{C}^n) : X_z \in e^{i(1-k)\pi/3}\mathbb{R}^n \text{ for } z \in \partial_k S\},$$

$$\mathscr{W}_{u,J}^0 = L^p(\Lambda^{0,1} S \otimes \mathbb{C}^n). \tag{3.8}$$

In (2.5) take $\nabla = \nabla^S \times \nabla^M$ to be the product of torsion-free connections on S and on M. Then the second term $(\nabla_X J) \circ Du \circ j$ in the formula vanishes, because $Du \circ j$ takes values in TE^{h} whereas $\nabla_X J$ is nontrivial only on TE^{v}; and moreover, the pullback connection $u^*\nabla$ on u^*TE^{v} is trivial. This shows that $D_{u,J}$ is the standard $\bar{\partial}$-operator for functions $S \to \mathbb{C}^n$, with boundary conditions (3.8).

There is a general index formula for such operators, but we prefer to use an ad hoc gluing instead. Consider the compact surface $\bar{S}$ in Fig. 4, which is of genus one with one boundary component. Parametrize the boundary by a closed path $l : [0; 6] \to \partial\bar{S}$, such that $l(t) = z_t$ for $t \in \{0, 1, 2, 3, 4, 5\}$ are the marked points in Fig. 4. Take a smooth nondecreasing function $\lambda : [0; 6] \to \mathbb{R}$ such that

$$\lambda(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1, \\ 1/3, & 2 \leqslant t \leqslant 3, \\ 2/3, & 4 \leqslant t \leqslant 5. \\ 1, & t = 6. \end{cases}$$

Let $\bar{D}$ be the $\bar{\partial}$-operator on the trivial bundle $\bar{S} \times \mathbb{C}^n \to \bar{S}$, with boundary condition given by the family of Lagrangian subspaces $\Lambda_{l(t)} = \mathrm{e}^{\pi \mathrm{i}\lambda(t)}\mathbb{R}^n \subset \mathbb{C}^n$. As a loop in the Lagrangian Grassmannian, this represents n times the standard generator of the fundamental group. Riemann–Roch for compact surfaces with boundary therefore tells us that $\operatorname{ind} \bar{D} = n\chi(\bar{S}) + n = 0$. On the other hand, one can divide $\bar{S}$ into five pieces (two shaded and three unshaded ones) as indicated in Fig. 4, and add strip-like ends to each piece; the standard gluing theory for elliptic operators says that $\operatorname{ind} \bar{D}$ is the sum of the indices of the obvious corresponding operators on those pieces. For each shaded piece, this yields a copy of $D_{u,J}$ (in one of the two cases, the vector space $\mathbb{C}^n$ has been rotated by $\mathrm{e}^{\pi\mathrm{i}/3}$). The unshaded pieces give rise to operators of index zero. This can be derived from the index theorem of [20], or else by directly deforming the operator to an invertible one. Consider for instance the two intervals of $\partial\bar{S}$ which are contained in the leftmost unshaded piece. The Lagrangian subspaces Λ_z parametrized by the points z in one of these intervals are all equal to $\mathrm{e}^{\pi\mathrm{i}/3}\mathbb{R}^n$; for the other interval, they are of the form $\mathrm{e}^{\pi\mathrm{i}s}\mathbb{R}^n$ for $\frac{2}{3} \leqslant s \leqslant 1$. Since $\mathrm{e}^{\pi\mathrm{i}/3}\mathbb{R}^n \cap \mathrm{e}^{\pi\mathrm{i}s}\mathbb{R}^n = 0$ for all such s, the Maslov index for paths [20] is zero. The same holds for the other unshaded pieces, and one concludes that $0 = \operatorname{ind} \bar{D} = 2 \operatorname{ind} D_{u,J}$. □

At this point, fix

$$J^{(1)} \in \mathscr{J}^{\mathrm{reg}}(M, \tau_L(L_0), L),$$

$$J^{(2)} \in \mathscr{J}^{\mathrm{reg}}(M, L, L_1),$$

$$J^{(3)} \in \mathscr{J}^{\mathrm{reg}}(M, \tau_L(L_0), L_1).$$

Proposition 2.28(i) tells us that there is a $J \in \mathscr{J}(E, \pi, j, J^{(1)}, J^{(2)}, J^{(3)})$ which is both horizontal and regular. By part (ii) of the same result, the coefficients $v_J(\tilde{x}_0, x_1, x)$ for $\tilde{x}_0 \in \tau_L(L_0) \cap L$, $x_1 \in L \cap L_1$, $x \in \tau_L(L_0) \cap L_1$, are zero whenever

$$\chi(\tilde{x}_0, x_1, x) = a_{\tau_L(L_0),L_1}(x) - a_{\tau_L(L_0),L}(\tilde{x}_0) - a_{L,L_1}(x_1) \tag{3.9}$$

is < 0. Lemma 3.2(ii) shows that $\chi(\tilde{x}_0, x_1, x)$, if $\geqslant 0$, must be either in $[0;\varepsilon)$ or $[3\varepsilon;\infty)$, and that the first case only happens for $x = p(\tilde{x}_0, x_1)$. Hence, if one writes the relative invariant as

$$C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J) = \phi + (C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J) - \phi),$$

$$\phi(\langle \tilde{x}_0 \rangle \otimes \langle x_1 \rangle) = v_J(\tilde{x}_0, x_1, p(\tilde{x}_0, x_1))\langle p(\tilde{x}_0, x_1)\rangle \tag{3.10}$$

then ϕ is of order $[0;\varepsilon)$, while the second term is of order $[3\varepsilon;\infty)$. The rest of this section contains the proof of the following result, which determines the low order part:

Proposition 3.4. $v_J(\tilde{x}_0, x_1, p(\tilde{x}_0, x_1)) = 1$ *for all* $(\tilde{x}_0, x_1)$.

There is one particular case of this which follows from the previous considerations. Namely, suppose that there is a pair $(\tilde{x}_0, x_1)$ with $\tilde{x}_0 = x_1 = x \in \tau_L(L_0) \cap L \cap L_1$. In that case $p(x, x) = x$ and $\chi(x, x, x) = 0$ by Lemma 3.2(iii), and therefore $v_J(x, x, x) = \#\mathscr{M}^{\mathrm{h}}(x, x, x)$ by Proposition 2.28(ii). Now $\#\mathscr{M}^{\mathrm{h}}(x, x, x) = 1$ because, as we saw already when proving Lemma 3.3, the unique element of that space is the constant section $u(z) = (z, x)$. Our strategy will be to reduce the computation of all the $v_J(\tilde{x}_0, x_1, p(\tilde{x}_0, x_1))$ to this case, by using suitable deformations.

What "deformation" means here is keeping the submanifolds L, L_0, L_1 and the constant ε fixed, while changing the remaining data. More precisely, for $0 \leqslant \mu \leqslant 1$ we consider the following: a smooth family f^μ of diffeomorphisms $S^n \to L$; smoothly varying positive numbers λ^μ, and symplectic embeddings $\imath^\mu : T(\lambda^\mu) \to M$ with $\imath^\mu | T(0) = f^\mu$; the Dehn twists τ_L^μ defined using $\imath^\mu$ and functions R^μ supported in $(-\infty; \lambda^\mu)$; and constants δ^μ. These should agree with the given data $f, \lambda, \imath, R, \tau_L, \delta$ for $\mu = 0$. We also require that the analogues of (III)–(V) continue to hold for any μ; where in (V) we take the original ε throughout.

Choose $\tilde{x}_0 \in \tau_L(L_0) \cap L$, $x_1 \in L \cap L_1$. Given a "deformation" in the sense which we have just explained, one can set $\tilde{x}_0^\mu = \tau_L^\mu (\tau_L)^{-1}(\tilde{x}_0) \in \tau_L^\mu(L_0) \cap L$ and $x_1^\mu = x_1 \in L \cap L_1$, which "continues" the points smoothly into the deformed situation. We claim that $x = p(\tilde{x}_0, x_1)$ fits similarly into a smooth family $x^\mu \in \tau_L^\mu(L_0) \cap L_1$. The point is that since (I)–(V) continue to hold, Lemma 3.2 can be applied to the situation for any μ. This ensures that the intersections $\tau_L^\mu(L_0) \cap L_1$ remain transverse, which implies that a unique family x^μ exists. In fact, it even provides a smooth family of injective maps $p^\mu : (\tau_L^\mu(L_0) \cap L) \times (L \cap L_1) \to \tau_L^\mu(L_0) \cap L_1$, such that $x^\mu = p^\mu(\tilde{x}_0^\mu, x_1^\mu)$.

Lemma 3.5. *For any $(\tilde{x}_0, x_1)$ there is a "deformation" such that $\tilde{x}_0^1 = x_1^1$.*

Proof. From (III) we know that $y_0 = f^{-1}(\tau_L^{-1}(\tilde{x}_0))$, $y_1 = f^{-1}(x_1)$ are points on S^n whose distance is $\geqslant 2\pi\delta$. Let $g^\mu \in \mathrm{Diff}(S^n)$ be an isotopy, $g^0 = \mathrm{id}$, such that $g^1(y_0)$, $g^1(y_1)$ are antipodes. There are $C^\mu \geqslant 1$, smoothly depending on μ and with $C^0 = 1$, with the property that

$$C^\mu \geqslant \|D(g^\mu)_y\|, \ \|D(g^\mu)_y^{-1}\| \quad \text{for all } y \in S^n.$$

Consider the "deformation" $f^\mu = f \circ (g^\mu)^{-1}$, $\lambda^\mu = \lambda / C^\mu$, $\imath^\mu = \imath \circ G^\mu | T(\lambda^\mu)$, $\delta^\mu = \delta / C^\mu$; here $G^\mu \in \mathrm{Symp}^e(T)$ is induced by g^μ, in the sense that $G^\mu | T(0) = (g^\mu)^{-1}$. The bound on $D(g^\mu)$ implies that G^μ maps $T(\lambda^\mu)$ to $T(\lambda)$, so that $\imath^\mu$ is well-defined. On the other hand, because of the bound on $D(g^\mu)^{-1}$, the distance between any point of $(f^\mu)^{-1}(L \cap L_0) = g^\mu f^{-1}(L \cap L_0)$ and any point of $(f^\mu)^{-1}(L \cap L_1) = g^\mu f^{-1}(L \cap L_1)$ is $\geqslant 2\pi\delta^\mu$, which shows that (III) holds during the deformation. For (IV) it is sufficient to note that G^μ takes the canonical one-form θ_T to itself and maps fibres to fibres. And it is no problem to find functions R^μ which satisfy (V) with the δ^μ defined above and the given ε. By definition

$$\tau_L | L = f \circ A \circ f^{-1}, \quad \tau_L^\mu | L = f^\mu \circ A \circ (f^\mu)^{-1} = f \circ (g^\mu)^{-1} \circ A \circ g^\mu \circ f^{-1}.$$

Since $g^1(y_1) = A(g^1(y_0))$ by construction, one sees that

$$\tilde{x}_0^1 = \tau_L^1(\tau_L)^{-1}(\tilde{x}_0) = f \circ (g^1)^{-1} \circ A \circ g^1(y_0) = f(y^1) = x_1 = x_1^1. \qquad \square$$

Given a "deformation", one can repeat the construction at the beginning of this section in a parametrized sense, which means equipping the trivial exact symplectic fibration (E, π) with a smooth family Q^μ of Lagrangian boundary conditions, modelled over the ends on the pairs of exact Lagrangian submanifolds $(\tau_L^\mu(L_0), L)$, (L, L_1), and $(\tau_L^\mu(L_0), L_1)$. Since their intersections are transverse for all μ, it makes sense to consider parametrized moduli spaces of pseudo-holomorphic sections. To do that, take smooth families of almost complex structures $J^{(1),\mu}, J^{(2),\mu}, J^{(3),\mu} \in \mathscr{J}(M)$ which reduce to the previously chosen ones for $\mu = 0$, and similarly a family $J^\mu \in \mathscr{J}^{\mathrm{h}}(E, \pi, j, J^{(1),\mu}, J^{(2),\mu}, J^{(3),\mu})$.

The parametrized moduli space we are interested in is

$$\mathscr{M}^{\mathrm{para}} = \bigcup_{\mu} \{\mu\} \times \mathscr{M}_{J^\mu}(\tilde{x}_0^\mu, x_1^\mu, x^\mu)$$

for points $\tilde{x}_0^\mu, x_1^\mu, x^\mu$ as introduced above.

Lemma 3.6. *$\mathscr{M}^{\mathrm{para}}$ is compact.*

Proof. Consider $CF(L, L_1)$, $CF(\tau_L^\mu(L_0), L)$, $CF(\tau_L^\mu(L_0), L_1)$. We already know that the first of these $\mathbb{R}$-graded vector spaces has gap $(0; 3\varepsilon)$; for the second one, the same is true because it agrees with $CF(L_0, L)$ up to a constant shift in the grading; and by repeating the considerations after Lemma 3.2, one can show that $CF(\tau_L^\mu(L_0), L_1)$ has gap $(0; 2\varepsilon)$. As another consequence of Lemma 3.2(ii) in the deformed situation, any $(\mu, u) \in \mathscr{M}^{\mathrm{para}}$ satisfies

$$A(u) = a_{\tau_L^\mu(L_0), L_1}(x^\mu) - a_{\tau_L^r(L_0), L}(\tilde{x}_0^\mu) - a_{L, L_1}(x_1^\mu) \in [0; \varepsilon).$$

One can now argue exactly as in Lemma 2.24; there are no points at infinity in the parametrized Gromov–Floer compactification, since the principal component of any such point would have negative action, which would contradict the fact that (E, π) has nonnegative curvature. □

Suppose now that the $J^{(k),\mu}$ and J^μ for $\mu = 1$ have been chosen regular; for J^1 this can be achieved without leaving the class of horizontal almost complex structure, because Lemma 3.3 equally applies to the deformed situation for $\mu = 1$. By using a parametrized version of the same lemma and of Proposition 2.28(i), one sees that in addition, the family (J^μ) can be chosen to be regular in the parametrized sense. Then the one-dimensional part of $\mathscr{M}^{\mathrm{para}}$ is a compact one-manifold, and its boundary points are precisely those with $\mu = 0$ or 1. One concludes that

$$v_J(\tilde{x}_0, x_1, x) = v_{J^1}(\tilde{x}_0^0, x_1^1, x^1).$$

If the "deformation" is as in Lemma 3.5, the situation for $r = 1$ is exactly the special case which we have already discussed, so that $v_{J^1}(\tilde{x}_0^0, x_1^1, x^1) = 1$. Since such deformations exist for all $(\tilde{x}_0, x_1)$, Proposition 3.4 is proved.

From now on, we fix some $J = J^{(4)} \in \mathscr{J}^{\mathrm{reg,h}}(E, \pi, Q, j, J^{(1)}, J^{(2)}, J^{(3)})$ and write $b = C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J^{(4)})$: $C' \to C$ for the relative invariant on the cochain level. Let β be the map defined at the end of Section 3.1. What (3.10) and Proposition 3.4 say is that $b = \beta + (b - \beta)$, with $b - \beta$ of order $[3\varepsilon; \infty)$, which is even slightly more than required by Lemma 2.32.

Remark 3.7. Our relative invariant is the well-known pair-of-pants product, or Donaldson product; in fact, if $u(z) = (z, \sigma(z))$ lies in $\mathscr{M}_J(\tilde{x}_0, x_1, x)$, then $\sigma : S \to M$ is a "pseudo-holomorphic triangle" whose sides lie on $\tau_L(L_0), L, L_1$ and whose vertices are $\tilde{x}_0, x_1, x$. Proposition 3.4 asserts that there is an odd number of low-area triangles with certain specified vertices. In the lowest dimension, $n = 1$, one can see directly that there is precisely one such triangle (Fig. 5). In higher dimensions it is still easy to construct explicitly the analogue of this particular triangle, but proving that there are no others seems more difficult. For that reason, we have preferred to take the indirect approach via "deformations", which effectively meant moving $\tau_L(L_0)$ in such a way that the area of the triangle shrinks to zero.

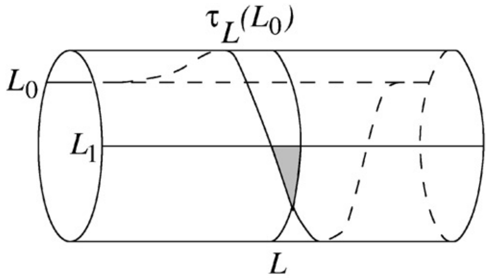

Fig. 5.

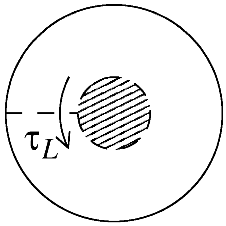

Fig. 6.

3.3. The second map

From this point onwards, we add one more assumption to those in Section 3.1:

(VI) R is the function R_r which appears in the construction of exact Lefschetz fibrations in Lemma 1.10, with the given λ and some $0 < r < \frac{1}{2}$ which we are free to choose; see more specifically (1.19) for this function.

Since that severely restricts the choice of R, one needs to worry about possible conflicts with the other conditions, or that it might restrict the ultimate scope of the exact sequence. An inspection of Remark 3.1 shows that the only issue is whether, using $R = R_r$, one can satisfy (V) with arbitrarily small δ and ε. That is taken care of by Lemma 1.12(iv), which shows that it suffices to choose r small.

The obvious reason for introducing (VI) is that there is a standard fibration (E^L, π^L) over a disc $\bar{D}(r)$, with r small, whose monodromy around $\partial\bar{D}(r)$ is τ_L. Following Remark 1.4, we want to modify this by a pullback. Take $S^p = \bar{D}(\frac{1}{2})$ and a map $p : S^p \to \bar{D}(r)$ of the form $p(z) = g(|z|)(z/|z|)$, where g is a function with $g(t) = t$ for small t, $g(t) = r$ for $t \geqslant r$, and $g'(t) \geqslant 0$ everywhere. Then

$$(E^p, \pi^p) = p^*(E^L, \pi^L)$$

is again an exact Lefschetz fibration. It has nonnegative curvature; this follows from Lemma 1.12(iii) and the fact that $\det(Dp) \geqslant 0$. Moreover, it is flat on the annulus $S^p \setminus D(r)$; and using the isomorphism $\phi^p : (E^p)_{1/2} \to M$ inherited from $\phi^L : (E^L)_r \to M$, one can identify its monodromy around ∂S^p with τ_L. To represent this property, we draw (E^p, π^p) as in Fig. 6.

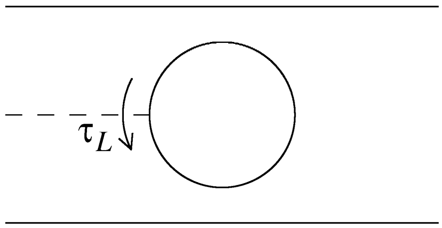

Fig. 7.

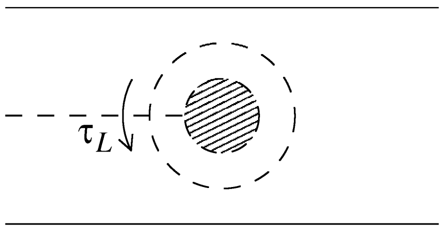

Fig. 8.

Now take the surface $S^f=(\mathbb{R}\times[-1;1])\setminus D(\frac{1}{2})\subseteq\mathbb{R}^2$, with coordinates (s,t), and divide it into two parts $S^{f,\pm}=S^f\cap\{t\in\mathbb{R}^\pm\}$, so that $S^{f,+}\cap S^{f,-}=((-\infty;-\frac{1}{2}]\cup[\frac{1}{2};\infty))\times\{0\}$. Consider trivial fibrations $\pi^{f,\pm}:E^{f,\pm}=S^{f,\pm}\times M\to S^{f,\pm}$ over the two parts, and equip them with differential forms $\Omega^{f,\pm}$, $\Theta^{f,\pm}$ as follows. $\Omega^{f,\pm}$ is the pullback of $\omega\in\Omega^2(M)$, and similarly $\Theta^{f,+}$ is the pullback of θ; finally $\Theta^{f,-}=\theta-\mathrm{d}(\beta(s)K_{\tau_L})$, where β is a function with $\beta(s)=0$ for $s\geqslant\frac{1}{4}$, $\beta(s)=1$ for $s\leqslant-\frac{1}{4}$. Define a fibration (E^f,π^f) over S^f by identifying the fibres

$$E^{f,+}_{(s,0)}\to E^{f,-}_{(s,0)}$$

via id_M for $s\geqslant\frac{1}{2}$, respectively, via τ_L for $s\leqslant-\frac{1}{2}$. $\Omega^{f,\pm}$ and $\Theta^{f,\pm}$ match up to forms Ω^f, Θ^f; the first because τ_L is symplectic, and the second as a consequence of our choice of $\Theta^{f,-}$. This makes (E^f,π^f) into a (flat) exact symplectic fibration; it is represented in Fig. 7.

We now carry out a pasting construction of the kind discussed in Section 1.1. If one identifies the fibres of E^p and E^f at the point $\frac{1}{2}$ by using $\phi^p:(E^p)_{1/2}\to M=(E^f)_{1/2}$, then the monodromies around the circle $|z|=\frac{1}{2}$ coincide, being both equal to τ_L. Since the two fibrations are flat close to this circle, one can paste them together to an exact Lefschetz fibration, denoted by (E,π), over $S=S^p\cup S^f=\mathbb{R}\times[-1;1]$; this is drawn in Fig. 8. Equip (E,π) with the Lagrangian boundary condition Q which is the union of $\mathbb{R}\times\{1\}\times L_1\subset E^{f,+}$ and $\mathbb{R}\times\{-1\}\times\tau_L(L_0)\subset E^{f,-}$, with $\kappa_Q=0$, and with a function K_Q which is K_{L_1} on $\mathbb{R}\times\{1\}\times L_1$ and $K_{\tau_L(L_0)}-\beta(s)K_{\tau_L}|\tau_L(L_0)$ on $\mathbb{R}\times\{-1\}\times\tau_L(L_0)$. This is clearly modelled on $(\tau_L(L_0),L_1)$ over the positive end of S; over the negative end it is modelled on (L_0, L_1), as one sees using the trivialization

$$\begin{tikzcd} (-\infty;-1] \cdot [-1;1] \cdot M \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "\pi"] \\ (-\infty;-1] \cdot [-1;1] \arrow[r, "\gamma"] & S \end{tikzcd}$$

$$\gamma = \text{inclusion}, \qquad \Gamma(s,t,x) = \begin{cases} (s,t,x) \in E^{f,+} & t \geqslant 0, \\ (s,t,\tau_L(x)) \in E^f, & t \leqslant 0. \end{cases}$$

We now get a relative invariant $\Phi_0^{\mathrm{rel}}(E,\pi,Q) : \mathrm{HF}(\tau_L(L_0),L_1) \to \mathrm{HF}(L_0,L_1)$. Following the same pattern as in the previous section (but with considerably less technical difficulties), we need to determine partially the underlying cochain map.

Lemma 3.8. *(E,π,Q) together with $U=(-1;1)\times[-1;1] \subset S$ satisfies the conditions of Proposition* 2.28(i). *Moreover, given $w \in \tau_L(L_0) \cap L_1$ and $x \in L_0 \cap L_1$, the space $\mathscr{M}^{\mathrm{h}}(x,w)$ contains precisely one horizontal section if $w = q(x)$, and is empty otherwise.*

Proof. Because the two parts from which it is assembled have nonnegative curvature, so does (E,π). To any point $x \in L_0 \cap L_1$ one can associate a horizontal section $u^f : S^f \to E^f$ satisfying $u^f(\partial S) \subset Q$, which is defined by

$$u^f(s,t) = \begin{cases} (s,t,x) \in E^{f,+} & t \geqslant 0, \\ (s,t,\tau_L(x)) \in E^{f,-} & t \leqslant 0. \end{cases}$$

Conditions (I) and (IV) imply that $x \notin \mathrm{im}(\imath)$. By construction E^L contains a trivial part $\bar{D}(r) \times (M \setminus \mathrm{im}(\imath))$, see Proposition 1.11. The pullback E^p has a corresponding property, which means that there is a horizontal section u^p of E^p, matching up with u^f to form a $u \in \mathscr{M}^{\mathrm{h}}(x,x) \subset \mathscr{M}^{\mathrm{h}}$.

$\int_{S^f}(u^f)^*\Omega^f = 0$ by definition of Ω^f, and similarly $\int_{S^p}(u^p)^*\Omega^p = 0$ because the image of u^p lies in the trivial part of E^p. It follows that the section we have constructed satisfies $A(u) = 0$. The connection ∇^u on u^*TE^{v} is trivial; in fact, by inspecting the details of the construction, one can see that (E,π) is symplectically trivial in a neighbourhood of $\mathrm{im}(u)$. By linearization of the basic fact that symplectic parallel transport preserves any Lagrangian boundary condition, one finds that the subbundle $u^*(TQ \cap TE^{\mathrm{v}}) \subset u^*TE^{\mathrm{v}}|\partial S$ is preserved under ∇^u; hence it is trivial. To summarize, we have found that one can identify $u^*(TE^{\mathrm{v}}) \cong S \times \mathbb{C}^n$ symplectically, in such a way that ∇^u becomes trivial, and that $u^*(TQ \cap TE^{\mathrm{v}})$ is mapped to the subbundle $(\mathbb{R} \times \{-1\} \times \Lambda_{-1}) \cup (\mathbb{R} \times \{1\} \times \Lambda_1)$ for some Lagrangian subspaces $\Lambda_{\pm 1} \subset \mathbb{C}^n$; by looking at the positive end, one sees that Λ_{-1} and Λ_1 are transverse. Then the index formula in terms of the Maslov index for paths [20] shows that $\mathrm{ind}\, D_{u,J} = 0$.

Next, suppose that $u \in \mathscr{M}^{\mathrm{h}}$ is an arbitrary horizontal section satisfying $u(\partial S) \subset Q$. When restricted to $S^{f,\pm}$, this is of the form $u^{f,\pm}(z) = (z, x^{\pm})$ for points $x^+ \in L_1$, $x^- \in \tau_L(L_0)$. The condition for the two parts to match along $S^{f,+} \cap S^{f,-}$ is that $x^- = x^+ = \tau_L(x^+)$. In particular $x^{\pm} \in L_0 \cap L_1$. Because of the strong unique continuation property of horizontal section it follows that the construction above

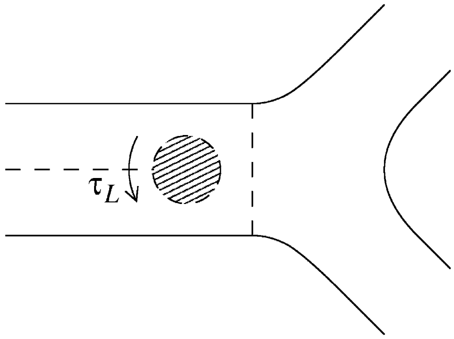

Fig. 9.

yields all of $\mathscr{M}^{\mathrm{h}}$. By the same argument, any partial horizontal section $U \to E|U$ with boundary in Q can be extended to some element of $\mathscr{M}^{\mathrm{h}}$. □

Let j be some complex structure on S, standard over the ends. Take the same $J^{(3)} \in \mathscr{J}^{\mathrm{reg}}(M, \tau_L(L_0), L_1)$ as in the previous section and choose an additional $J^{(5)} \in \mathscr{J}^{\mathrm{reg}}(L_0, L_1)$. By Lemma 3.8 and Proposition 2.28, one can find a $J^{(6)} \in \mathscr{J}(E, \pi, Q, j, J^{(3)}, J^{(5)})$ which is both horizontal and regular. Write

$$c = C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J^{(6)}) : C \to C''$$

for the chain map defined by this. In view of Lemmas 2.29 and 3.2(i) one can write $c = \phi + (c - \phi)$, where ϕ depends only on the section in $\mathscr{M}^{\mathrm{h}}$ and is of order $\{0\}$, and the remaining term is of order $[3\varepsilon; \infty)$. Again applying Lemma 3.8, one finds that ϕ is precisely the map γ defined at the end of Section 3.1. Again, this is marginally better than what is needed to apply Lemma 2.32.

3.4. The homotopy

At this point it becomes necessary to change the notation slightly, in order to avoid conflicts. Thus, the fibration used in Section 3.2 to define the map b will be denoted by (E^b, π^b), its base by S^b, and its Lagrangian boundary condition by Q^b; and correspondingly we write (E^c, π^c), S^c, Q^c for the objects constructed in Section 3.3 to define c. Over the unique negative end of S^b, Q^b is modelled on $(\tau_L(L_0), L_1)$, and the same holds for Q^c over the positive end of S^c. As described in Section 2.4, one can glue these ends together to obtain a new exact Lefschetz fibration (E^{bc}, π^{bc}) over a surface S^{bc}, with a Lagrangian boundary condition Q^{bc}. The outcome is represented schematically in Fig. 9.

(E^{bc}, π^{bc}) has nonnegative curvature because (E^b, π^b) and (E^c, π^c) have that property. Moreover $\mathscr{M}^{\mathrm{h}} = \emptyset$, which means that there are no horizontal section $u : S^{bc} \to E^{bc}$ with $u(\partial S^{bc}) \subset Q^{bc}$. In fact, assuming that such a u exists, one could reverse the gluing construction and obtain a horizontal section u^b of E^b with boundary in Q^b, as well as a similar section u^c of E^c. We have seen previously that such u^b correspond to points in $\tau_L(L_0) \cap L \cap L_1$, and u^c to points in $L_0 \cap L_1$. In our case, the two section would have to match over the ends used in the gluing process, which means that they would correspond to a point of $L_0 \cap L \cap L_1$; but that is impossible by (I). The same argument shows that for a sufficiently large relatively compact subset $U \subset S^{bc}$, there are no horizontal $w : U \to E^{bc}$ with

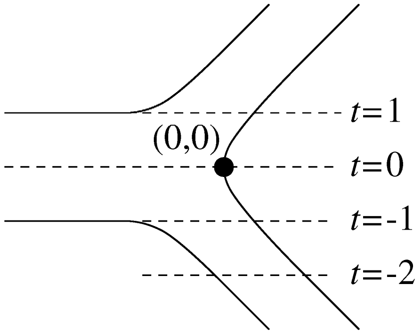

Fig. 10.

$w(\partial S^{bc} \cap U) \subset Q^{bc}$. Over the ends, Q^{bc} is modelled on $(\tau_L(L_0), L)$, (L, L_1) and (L_0, L_1), respectively. For these pairs of submanifolds we have already chosen almost complex structures $J^{(1)}$, $J^{(2)}$ and $J^{(5)}$, respectively. Let j be some complex structure on S^{bc} which is standard over the ends. Lemma 2.28 ensures that one can choose a $J \in \mathscr{J}(E^{bc}, \pi^{bc}, Q^{bc}, j, J^{(1)}, J^{(2)}, J^{(5)})$ which is both horizontal and regular.

Lemma 3.9. *For any such* J,

$$C\Phi_0^{\mathrm{rel}}(E^{bc}, \pi^{bc}, Q^{bc}, J) : C' \to C''$$

is homotopic to $c \circ b$ *by a chain homotopy which is of order* $(0; \infty)$.

Proof. Suppose first that j is induced from the complex structures on S^b, S^c, and that J is similarly constructed from $J^{(4)}$ and $J^{(6)}$; this is automatically horizontal. Proposition 2.22 says that for large values of the gluing parameter σ, J is regular; and then we have moreover $C\Phi_0(E^{bc}, \pi^{bc}, Q^{bc}, J) = c \circ b$ by (2.30). On the other hand, the observations made above allow us to apply Lemma 2.30, which shows that the maps $C\Phi_0$ for any two choices of j and J are homotopic by a chain homotopy of order $(0; \infty)$. □

The proof that $c \circ b$ is a chain homotopic to zero relies on an alternative construction of the same exact Lefschetz fibration. Consider the surface S^0 from Fig. 10, embedded into $\mathbb{R}^2$ with coordinates (s, t). As in the previous section we divide it into $S^{o,\pm} = S^o \cap \{t \in \mathbb{R}^\pm\}$ and take the trivial fibrations $\pi^{o,\pm}$: $E^{o,\pm} = S^{o,\pm} \times M \to S^{o,\pm}$. We equip $E^{o,\pm}$ with the forms $\Omega^{o,\pm}$ pulled back from ω, and $E^{o,+}$ with the one-form $\Theta^{o,+}$ pulled back from θ; while on $E^{o,-}$ we take $\Theta^{o,-} = \theta - \mathrm{d}(\eta(t) K_{\tau_L})$, where η is some function with $\eta(t) = 1$ for $t \geqslant -1$, $\eta(t) = 0$ for $t \leqslant -2$. One now identifies the fibres over $z \in \mathbb{R}^- \times \{0\} = S^{o,+} \cap S^{o,-}$ by using $\tau_L : E_z^{o,+} \to E_z^{o,-}$, which yields an exact symplectic fibration (E^o, π^o) over S^o (it is in fact trivial, but for us it is convenient to think of it as being built up in this particular way). Take the pieces of ∂S^o labeled in Fig. 11, and construct a Lagrangian boundary condition Q^o for (E^o, π^o) as the union of $\partial_1 S^{o,-} \times \tau_L(L_0), \partial_2 S^{o,-} \times L \subset E^{o,-}$ and $\partial_2 S^{o,+} \times L$, $\partial_3 S^{o,+} \times L_1 \subset E^{o,+}$. The associated function K_{Q^o} is equal to $K_{\tau_L(L_0)} - \eta(t) K_{\tau_L} | \tau_L(L_0)$ over $\partial_1 S^{o,-}$, to K_{L_1} over $\partial_3 S^{o,+}$, and zero on the rest. κ_{Q^o} is equal to $-(K_{\tau_L}|L)\eta'(t)\,\mathrm{d}t$ on $\partial_2 S^{o,-}$ and vanishes elsewhere; this makes sense because $K_{\tau_L}|L$ is constant, equal to $-2\pi R(0)$ by (3.2). Combining this with (V) shows

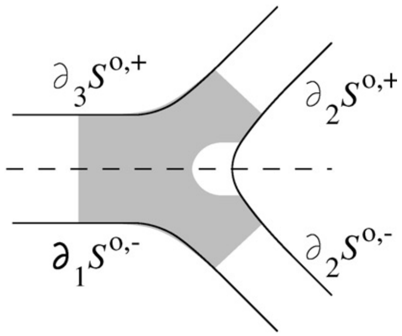

Fig. 11.

that

$$\int_{\partial S^o} \kappa_{Q^o} = 2\pi R(0) \in (-\varepsilon; 0]. \tag{3.11}$$

Q^o is modelled on $(\tau_L(L_0), L)$ and (L, L_1) over the positive ends, and on (L_0, L_1) over the negative end; a suitable trivialization of (E^o, π^o) over that end can be defined in the same way as in Section 3.3. The next statement is a straightforward consequence of (I):

Lemma 3.10. *Take the subset $U^o \subset S^o$ shaded in Fig. 11. Then there are no horizontal sections $u : U^o \to E^o$ with $u(\partial S^o \cap U^o) \subset Q^o$.*

For the following step, we need the fibration (E^p, π^p) over $S^p = \bar{D}(\frac{1}{2})$ from Section 3.3, which was defined as pullback of a standard fibration. The standard boundary condition from Section 2.3 pulls back to a Lagrangian boundary condition Q^p for it. As it stands $\kappa_{Q^p} = d^c(-\frac{1}{4}|z|)$ is nonzero everywhere, but for us it is better to modify it by some exact one-form, in such a way that it becomes zero near $-\frac{1}{2} \in \partial S^p$. This can be compensated by a change of K_{Q^p}, so that the whole remains a Lagrangian boundary condition. It is a consequence of Proposition 2.13 that the Φ_1-invariant of (E^p, π^p, Q^p) vanishes, since this can be joined to (E^L, π^L, Q^L) by a smooth deformation of exact Lefschetz fibrations with Lagrangian boundary conditions.

Lemma 3.11. *Let $U^p \subset S^p$ be the complement of a sufficiently small neighbourhood of $-\frac{1}{2} \in \partial S^p$. Then there are no partial horizontal section $w : U^p \to E^p$ satisfying $w(\partial S^p \cap U^p) \subset Q^p$.*

Proof. From the standard fibration, E^p inherits a smooth family of Lagrangian spheres $\Sigma_z^p \subset E_z^p$, $z \neq 0$. These are carried into each other by parallel transport along any path, and they degenerate to the critical point $x_0 \in E_0^p$ as $z \to 0$. Now our boundary condition is made up of the Σ_z^p for $z \in \partial S^p$; therefore a section w with the properties stated above would satisfy $w(z) \in \Sigma_z^p$ for all $z \in \partial S^p \cap U^p$, and by parallel transport for all $z \in U^p \neq \{0\}$. In the limit this yields $w(0) = x_0$, but that is impossible since x_0 is a critical point. □

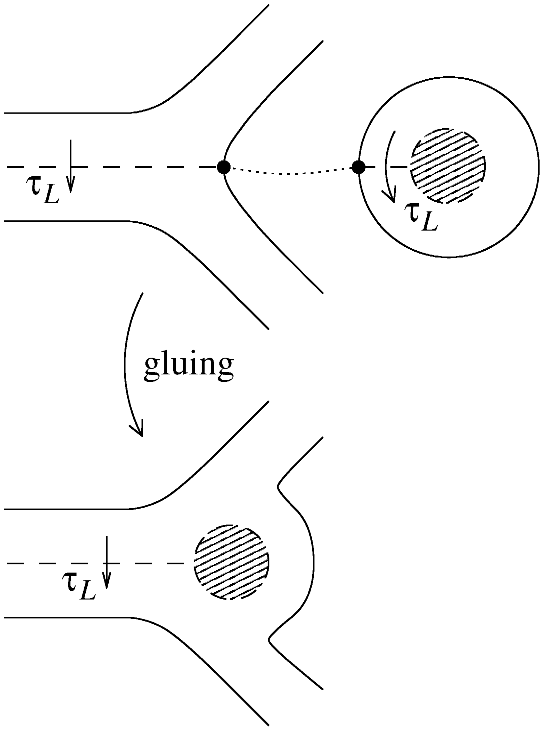

Fig. 12.

One can identify the fibre of E^o over $\zeta^o = (0,0) \in \partial S^o$ with the fibre of E^p over $\zeta^p = -\frac{1}{2} \in \partial S^p$ via

$$(E^o)_{\zeta^o} \cong (E^{o,+})_{(0,0)} = M \xleftarrow{\phi^p} E^p_{1/2} \cong (E^p)_{\zeta^p}, \tag{3.12}$$

where the last isomorphism is parallel transport along the upper semi-circle $(\frac{1}{2})e^{it}$, $0 \leqslant t \leqslant \pi$. By putting together the various definitions, one sees that (3.12) takes $(Q^o)_{(0,0)}$ to $(Q^p)_{-1/2}$. The fibrations (E^o, π^o) and (E^p, π^p) are flat near the points ζ^o, ζ^p, and the one-forms $\kappa_{Q^o}, \kappa_{Q^p}$ vanish near those points. This allows one to use the gluing construction discussed in Section 2.1 and again in Section 2.4 to produce an exact Lefschetz fibration (E, π) over $S = S^o \#_{\zeta^o \sim \zeta^p} S^p$, together with a Lagrangian boundary condition Q; see Fig. 12. Of course, this is still modelled over the ends on the same Lagrangian submanifolds as S^o. In what follows, we assume that the parameter ρ in the gluing process has been chosen sufficiently small.

Lemma 3.12. *There is a complex structure j on S, standard over the ends, and a $J \in \mathscr{J}^{\mathrm{reg,h}}(E, \pi, Q, j, J^{(1)}, J^{(2)}, J^{(5)})$, such that*

$$C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J) : C' \to C''$$

is homotopic to zero by a chain homotopy of order $[-2\pi R(0); \infty)$.

Proof. Let j^o be some complex structure on S^o, standard over the ends. Using Lemmas 3.10 and 2.28 one can find a $J^o \in \mathscr{J}(E^o, \pi^o, j^o, J^{(1)}, J^{(2)}, J^{(5)})$ which is horizontal and regular. In fact, regularity can be achieved even while prescribing what J^o is outside $(\pi^o)^{-1}(U^o) \subset E^o$. This is useful because, for our intended gluing argument, J^o needs to be of a particular form close to the fibre over $\zeta^o \notin U^o$: namely, in a local trivialization near that point, it needs to be the product of j^o and some previously fixed almost complex structure on M. We now have evaluation maps, for $\tilde{x}_0 \in \tau_L(L_0) \cap L$, $x_1 \in L \cap L_1$, $x \in \tau_L(L_0) \cap L_1$,

$$\mathrm{ev}_{\zeta^o}\colon \mathscr{M}_{J^o}(\tilde{x}_0, x_1, x) \to Q^o_{\zeta^o}. \tag{3.13}$$

Take some complex structure j^p on S^p. One can use Lemmas 2.9 and 3.11 to find a $J^p \in \mathscr{J}^{\mathrm{reg,h}}(E^p, \pi^p, Q^p, j^p)$ with fixed behaviour outside $(\pi^p)^{-1}(U^p)$; as before, since $\zeta^p \notin U^p$, one can use this to make J^p suitable for gluing. At the same time, Lemma 2.10 allows us to make the evaluation map $\mathrm{ev}_{\zeta^p} : \mathscr{M}_{J^p} \to Q^p_{\zeta^p}$ transverse to any given cycle. We take this cycle to be the disjoint union of (3.13) for all $\tilde{x}_0, x_1, x$, identifying $Q^o_{\zeta^o}$ and $Q^p_{\zeta^p}$ via (3.12).

Let j be the complex structure on S glued together from j^o, j^p, and similarly $J \in \mathscr{J}^{\mathrm{h}}(E, \pi, Q, j, J^{(1)}, J^{(2)}, J^{(5)})$ the almost complex structure obtained from J^o and J^p. As discussed in Section 2.4, J will be regular if the gluing parameter has been chosen sufficiently small. Moreover, the zero-dimensional spaces of (j,J)-holomorphic sections can be described as fibre products of those on both parts of the gluing, as in (2.28). We know that $\Phi_1(E^p, \pi^p, Q^p)$ is zero (as pointed out in Remark 2.13(i), this remains true even if we consider it as a cobordism class), and using Lemma 2.21 one concludes that $C\Phi_0^{\mathrm{rel}}(E, \pi, Q, J)$ is homotopic to zero. Inspection of the proof of that lemma will show that the homotopy constructed there is of order $[-2\pi R(0); \infty)$. In fact, its coefficients are given by the number of points in fibre products

$$\mathscr{M}_{J^o}(\tilde{x}_0, x_1, x) \times_{Q^o_{\zeta^o}} G, \tag{3.14}$$

where (G, g) is some cycle bounding $(\mathscr{M}_{J^p}, ev_{\zeta^p})$. Using (3.11) and Lemma 2.23 one sees that (3.14) is empty unless $a_{L_0,L_1}(x) \geqslant a_{\tau_L(L_0),L}(\tilde{x}_0) + a_{L,L_1}(x_1) - 2\pi R(0)$, which provides the desired estimate. $\square$

To put together Lemmas 3.9 and 3.12, one observes that there are oriented diffeomorphisms

$$\begin{tikzcd} E^{bc} \arrow[r, "\Psi"] \arrow[d, "\pi^{bc}"'] & E \arrow[d, "\pi"] \\ S^{bc} \arrow[r, "\psi"] & S \end{tikzcd}$$

with the following properties: on the ends, ψ and Ψ relate the local trivializations of the two fibrations. Next, $\Psi^*\Omega = \Omega^{bc}$, and $\Psi(Q^{bc}) = Q$. Finally, ψ is holomorphic near the unique critical value, with respect to the complex structures defined there which are part of the structure of exact Lefschetz fibrations of (E^{bc}, π^{bc}) and (E, π); and the same holds for Ψ near the unique critical point. This is not difficult, since both fibrations contain a copy of (E^p, π^p) and are otherwise flat; comparing Figs. 9 and 12 shows how the bases should be identified in order for the monodromies to match. It follows that one can take j and J in Lemma 3.9 to be the pullback of almost complex structures from Lemma 3.12, and then $C\Phi_0^{\mathrm{rel}}(E^{bc}, \pi^{bc}, Q^{bc}, J)$ will be homotopic to zero by a chain homotopy of order $[-2\pi R(0); \infty)$, with $-2\pi R(0) > 0$. On the other hand, it is homotopic to $c \circ b$

by a homotopy of order $(0;\infty)$. Taking the two together, one has a homotopy $h : c \circ b \simeq 0$ of order $(0;\infty)$, thus fulfilling the final requirement of Lemma 2.32.

Acknowledgements

Much of this work was originally done in 1996. It goes without saying that I am heavily indebted to Simon Donaldson, who was my advisor at that time. Conversations with Michael Callahan, Mikhail Khovanov, Dietmar Salamon, and Richard Thomas have been helpful. A preliminary version of this paper was presented in a series of talks at Ecole Polytechnique in 1999; I thank my colleagues there for their patience and comments. The final touches were put on during a visit to the University of Michigan, which provided a hospitable environment.

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